Viscous Lattices

Definition

Connections

godesic Reconstruction

Application to watershed

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Bibliography

- **Initial Idea:**
  F. Meyer, La viscosité en ligne de partage des eaux. Note CMM 1994

- **Applications:**
  1- Ph. Degrize, Reconstruction d'images IRM ; Analyse automatique d'images par Morphologie Mathématique. Phd Thesis, Université Paris 7, 1994 ;
  2- C. Vachier, F. Meyer et R. Lamara : Segmentation d'image par simulation d'une inondation visqueuse. To be published in RFIA 2000 - Paris ;

- **Current Development:**
  J. Serra, Viscous Lattices Note CMM D01-2000.
Objectives (1)

Purpose: To swell the space so that the Lion be wedged in its cage.
Objectives (2)

- **Method**: To generate *viscous* reconstruction, *i.e.* with a given meniscus.

- **Means**:
  1. To build the convenient lattice $\Lambda$: it is called viscous, and generated by the dilates of the points by a given dilation $\delta$ (the viscosity);
  2. To define connections on $\Lambda$ and to use the associated connected openings.

- **An example**: Contour of the heart muscle.
Notation and Reminder

- **Notation**:
  
  \( E \): an arbitrary set; \( \Pi(E) \): the lattice of all subsets of \( E \);
  
  \( \delta \): dilation \( \Pi(E) \to \Pi(E) \), of adjoint erosion \( \partial \). Dilation \( \delta \) is determined by the images of the singletons \( \{ \cdot \} \) de \( \Pi(E) \):
  
  \[
  \delta(X) = \bigcap \{ \delta(\cdot), \cdot \in X \} \quad X \in \Pi(E) \quad (1);
  \]

  \( B = \{ \delta(\cdot), \cdot \in E \} \): class of the dilates of the singletons;
  
  \( \gamma = \delta \Delta \partial = \) opening adjoint to dilation \( \delta \).

- **Reminder**:
  The family \( \Lambda = \{ \delta(X), X \in \Pi(E) \} \) of the dilates of the elements of \( \Pi(E) \) is also the image of \( \Pi(E) \) under the opening \( \gamma = \delta \partial \), adjoint to dilation \( \delta \).
Proposition 1:

Set $\Lambda$ has a structure of complete lattice for the ordering of the inclusion. In this lattice, the supremum coincides with the set union, although the infimum $\wedge$ is the opening of the intersection by $\gamma = \delta \partial$

$$\wedge\{X_i, i \in I\} = \gamma(\cup\{X_i, i \in I\}) \quad \{X_i, i \in I\} \in \Lambda(2)$$

- The extreme elements of $\Lambda$ are $E$ and the empty set $\emptyset$.
- Set $\Lambda$ is said to be the viscous lattice of dilation $\delta$. 
Atoms et Sup-generators

• **Sup-generators** :
  The class \( \mathcal{B} \) of the singletons dilates is a sup-generator of lattice \( \Lambda \).

• **Atoms** :

  But the \( \delta(\mathfrak{L}) \) themselves are not atoms in general. However, when \( E = \nabla^n \) or \( \wedge^n \), and for \( \delta \) invariant under translation, the elements of \( \mathcal{B} \) are the translates of the transform \( B = \delta(0) \) of the origin.

  In such a case, and for \( B \) a compact set, the associated viscous lattice is atomic, of atoms the translates de \( B \).
• **Proposition 2:**
  The viscous lattice of dilation $\delta$ is generally neither distributive, nor co-prime, and does not admit unique complements.

• **An example of the lack of distributivity:**

\[
(X \land X') \cap Y = \emptyset 
eq (X \cap Y) \land (X' \cap Y) = (X \cap Y)
\]
\[
(X \cap X') \land Y' = Y' \neq (X \land Y') \cap (X' \land Y') = \emptyset
\]
A few Counter-performances (2)

**Complement:**
A complement to \( X \in \Lambda \) is every set \( Y \in \Lambda \) such that
\[
Y \in X^c \quad \text{and} \quad \gamma(Y \cup X) = \emptyset
\]

**An example:**

**What is left:**
is Galois involution between adjoint dilation and erosion, i.e.

\[
\delta(Y) \subseteq X \quad \Leftrightarrow \quad \delta(X) \subseteq Y \quad \text{for} \quad X, Y \in \Lambda \quad \text{(3)}
\]
Dilation and Erosion in $\Lambda$

• **Links between a dilation viewed in $\Lambda$ and in $\Pi(E)$:**

  \[
  \begin{align*}
  & \text{Dilation} & \text{erosion} \\
  & \alpha \in \Pi(E) \rightarrow \Pi(E) & \alpha^1 : \Pi(E) \rightarrow \Pi(E) \\
  & \alpha : \Lambda \rightarrow \Lambda & \beta : \Lambda \rightarrow \Lambda
  \end{align*}
  \]

  Identity of $\alpha$ acting in $\Pi(E)$ or in $\Lambda$...

• **Links between an erosion dilation viewed in $\Lambda$ and in $\Pi(E)$:**

  \[
  \beta(X) = \bigcap \{ \delta(X), \delta(X) \subseteq \alpha^{-1}(X) \} = \beta(X) = \delta \partial \alpha^{-1}(X)
  \]

  When $\alpha$ and $\delta$ commute, then erosion $\beta$, in $\Lambda$, equals the opening by $\gamma = \delta \partial$ of erosion $\alpha^{-1}$, adjoint to dilatation $\alpha$ in $\Pi((E))$. 
Connection (Reminder)

• **Set case:**
  Every set family $X \subseteq \Pi(E)$ that satisfies the 3 following axioms
  
  i / $\emptyset \in X$
  
  ii/ $\downarrow \in E \Rightarrow \{\downarrow\} \in X$ (class $X$ is sup generating)
  
  iii/ $\{X_i, i \in I\} \in X$ et $\cup X_i \neq \emptyset \Rightarrow \cap X_i \in X$ (class $X$ is conditionally closed under union).

• **Generalisation:**
  The definition of connectivity extends to any sup generating lattice, by replacing $\emptyset$ by the zero of the lattice, and the set $\cap$ and $\cup$ by the sup et l'inf of the lattice. Axiom ii/ just means that class $X$ is sup-generating.
First Connections on $\Lambda(1)$

**Definition:**
Let $\Lambda$ be a viscous lattice of dilation $\delta$ on $\Pi(E)$. A class $X'$ of $\Lambda$ defines a connection on $\Lambda$ when

1. $\emptyset \in X'$
2. $\tau E = \{\{\} \in X'$
3. $\{X_i, i \in I\} \in X'$ and $\bigwedge X_i \neq \emptyset \Rightarrow \bigcap X_i \in X'$

**Proposition 3:**
A class $X \subseteq \Lambda$ is a connection on $\Lambda$. if it is the restriction to $\Lambda$ of the union of a connection $X$ on $\Pi(E)$, and of the image $B$ of the singletons of $\Pi(E)$ under $\delta$, i.e. if

$$X' = (X \Lambda) \cup B = (X \cup B) \Lambda$$
First Connections on $\Lambda(2)$

- **An Example:**

  \[ X = \text{Arcwise connection} \quad \Lambda = \text{set of the dilates by the unit disc } B. \]

  The union of the three lobes belongs to both $X$ and $\Lambda$, hence to $X'$. However, the three lobes are disjoint in $\Lambda$ because their erosions by $B$ are disjoint in $\Pi(E)$.

  If we want to get separated particles here (for connections on $\Lambda$), we have to take into account the status of connectivity (in $\Pi(E)$) \textit{before} dilation by $\delta$. 
• **Theorem 1:**

Let $X$ be a connection on $\Pi(E)$ and $\delta : \Pi(E) \rightarrow \Pi(E)$ be a dilation, of adjoint erosion $\partial$, which generates the viscous lattice $\Lambda$. Then the image $X' = \delta(X)$ of connection $X$ turns out to be a connection on $\Lambda$.

• **Comments:**

• these new connections on $\Lambda$, are sensibly more restrictive than those of prop. 4.

• For example, the three lobes of the figure are now three disjoint connected components.
Second Connections on $\Lambda(2)$

- **Generality of the theorem:**

- We did not assume that the dilates of the elements de $X$ are still $X$-connected. For example, in $\nabla^2$, with the usual arcwise connection, take for $\delta$ the dilation by a doublet of points from $h$ apart. Then the left two discs of the figure form a connected set, although the group of the three discs is no longer connected.

- However, if $\varepsilon \in \varepsilon \Rightarrow \delta \{ \varepsilon \} \varepsilon \ X$, we have $X' = \delta (X) \subseteq X$ and the elements of $X'$ are then connected in both lattices $\Lambda$ and $\Pi(E)$: Dilation $\delta$ preserves connection $X$. 
Geodesic Operators (1)

• **Theorem 2 :**

  When dilation $\delta$ preserves connection $X$, then connection $X'$ induced on $\Lambda$, in the sense du theorem 1, is also preserved by $\delta$.

  In addition, if $\gamma_\perp$ et $\gamma_\perp'$ stand for the elementary connected openings on $\Pi(E)$ and on $\Lambda$ respectively, then

  $$\gamma_{\perp'} = \delta \partial \gamma_\perp = \gamma_\perp \delta \partial$$

• **Comment :**

  Every $X'$-particle is the opening by $\delta \partial$ of the corresponding $X$-particle. That allows to extract the $X'$-particles.

  Alternatively, can we build **directly** geodesic dilations in $\Lambda$ ?
Proposition 4:

Let dilation $\delta$ be extensive and preserving connection $X \subseteq \Pi((E))$. Given $A;Z \in \Pi((E))$, with $A \subseteq Z$ and $X$-connected, the conditional dilate

$$\zeta(A) = \delta(A) \wedge Z$$

is $X'$-connected and included in $Z$.

N. B.: This does not mean that the iterated versions of $\zeta$ tend to a $X'$-component of $Z$. Here is a counter-example in $\nabla^2$:

- $E :=$ square $D$ of side $a$ and of centre $y$;
- $B(y) :=$ disc of diameter $< a$ centred in $y$
- Dilation := $\delta(\downarrow) = \{\downarrow\} \downarrow \in D / y$; $\delta(y) = B(y)$
• **Definition:**

Dilation $\delta$ is said to be **completely extensive** when for any point $\downarrow \in E$ (E topological space) and for any compact set $K \subseteq E$, there exists an integer $n$ such that the $n^{th}$ iteration of $\delta(\downarrow)$ covers $K$.

• **Proposition 5:**

When dilation $\delta$ is completely extensive, then the $X'$-connected component $Z$ of $\Lambda$ marked by $\Lambda \in X'(\Lambda)$, if it can be covered by a compact set $Z_0$ of $\Pi(E)$, is equal to

$$Z = \zeta^{(n)}(A)$$

for some integer $n$. 
**Binary example**: in white, marker A; in black, complement of mask Z; dilation \( \delta \) is the Minkowski addition by the disc of radius \( r \):

- **a)** step of reconstruction for \( r = 15 \);
- **b)** reconstruction for \( r = 27 \);
- **c)** maximum reconstruction, corresponding to \( r = 17 \).
An Example of Binary Geodesy (2)

Binary example (followed):

a) maximum reconstruction from the edges of the field
b) partition of $\mathbb{Z}$ into two $X'$-connected components for $\delta_{\text{max}}$
c) corresponding median element.
Numerical example:

a) positron image of the heart muscle (copyright CEA-ARMINES);
b) Watershed line of the gradient of fig.a
c) optimum reconstruction of the $X'$-components internal to zone b.
Numerical example (followed):

- **a)** external surface of maximum viscosity;
- **b)** external and internal contours;
- **c)** median element between the two contours.