Connections for Sets and Functions

Invited Lecture given at ISS-98
Amsterdam, April 21st, 1998

Jean SERRA
Ecole des Mines de Paris
Connectivity in Mathematics

- **Topological Connectivity**: Given a topological space $E$, set $A \subseteq E$ is connected if one cannot partition it into two non-empty closed sets.

- **A Basic Theorem**: If $\{A_i\}_{i \in I}$ is a family of connected sets, then
  
  \[
  \bigcap_{i} A_i \neq \emptyset \implies \bigcup_{i} A_i \text{ connected}
  \]

- **Arcwise Connectivity** (more practical for $E = \mathbb{R}^n$): $A$ is arcwise connected if there exists, for each pair $a, b \in A$, a continuous mapping $\psi$ such that
  
  \[\alpha, \beta \in \mathbb{R} \quad \text{and} \quad f(\alpha) = a \; ; \; f(\beta) = b\]

  This second definition is more restrictive. However, for the open sets of $\mathbb{R}^n$, both definitions are equivalent.
Criticisms

Is topological connectivity adapted to Image Analysis?

• Digital versions of arcwise connectivity are extensively used:
  – in 2-D: 4- and 8- connectivities (square), or 6- one (Hexagon);
  – in 3-D: 6-, 12-, 26- ones (cube) and 12- one (cube-octaedron).

However:

• Planar sectioning (3-D objects) as well as sampling (sequences) tend to disconnect objects and trajectories, and topological connectivity does help so much for reconnecting them;

• More generally, in Image Analysis, a convenient definition should be operating, i.e. should introduce specific operations;

• Finally, the topological definition is purely set oriented, although it would be nice to express also connectivity for functions...
Lattices and Sup-generators

- A common feature to sets $\mathcal{P}(E)$ (E an arbitrary space) and to functions $f: E \rightarrow T$ (T, grey axis) is that both form complete lattice that are «well» sup-generated.

- A **complete lattice** $\mathcal{L}$ is a partly ordered set where every family $\{a_i\} i \in I$ of elements admits
  - a smaller upper bound $\vee a_i$, and a larger lower bound $\wedge a_i$.

- A family $\mathcal{B}$ in $\mathcal{L}$ constitutes a **sup-generating** class when each $a \in \mathcal{L}$ may be written $a = \vee \{b ; b \in \mathcal{B} , b \leq a \}$.

- In $\mathcal{P}(E)$ - $\vee$ and $\wedge$ operations become union and intersection;
  - the elements of E, *i.e.* the points, are sup-generators.
In order to avoid the continuous/digital distinction, the real lines $\mathbb{R}$ and $\mathbb{Z}$, or any of their compact subsets, are all denoted by $T$. Axis $T$ is a totally ordered lattice, of extreme elements 0 and $m$.

• The class of functions $f : E \to T$, $E$ an arbitrary space, forms a totally distributive lattice, denoted by $T^E$, for the product ordering $f \leq g$ iff $f(x) \leq g(x)$ for all $x \in E$.

In this lattice, the so called numerical $\lor$ and $\land$ are defined by:

\[(\lor f_i)(x) = \lor f_i(x) \text{ and } (\land f_i)(x) = \land f_i(x) \quad x \in E.\]

• Moreover, in $T^E$ the pulses functions:

\[k_{x,t}(y) = t \text{ when } x = y ; \quad k_{x,t}(y) = 0 \text{ when } x \neq y,\]

are sup-generating, i.e. every function $f$ is written as

\[f = \lor \{ k_{x,t} \ , \ x \in E, \ t \leq f(x) \} .\]
• **Reminder**: A **Partition** of space E is a mapping \( D: E \rightarrow \mathcal{P}(E) \) such that

\[
\begin{align*}
(i) & \quad \forall x \in E, \quad x \in D(x) \\
(ii) & \quad \forall (x, y) \in E, \\
\text{either} & \quad D(x) = D(y) \\
\text{or} & \quad D(x) \cap D(y) = \emptyset
\end{align*}
\]

• The partitions of E form a **lattice** \( \mathcal{D} \) for the ordering in which \( D \leq D' \) when each class of \( D \) is included in a class of \( D' \). The largest element of \( \mathcal{D} \) is E itself, and the smallest one is the pulverization of E into all its points.

The sup of the two types of cells is the pentagon where their boundaries coincide. The inf, simpler, is obtained by intersecting the cells.
Connections on a Lattice

Since the basic property of topological connectivity involves set $\cup$ and $\cap$ only, we can forget all about topology and take the basic property, expressed in the lattice framework, as a starting point.

**Connection**: Let $L$ be a complete lattice. A class $C \subseteq L$ defines a connection on $L$ when

(i) $0 \in C$ ;

(ii) $C$ is sup-generating ;

(iii) $C$ is conditionally closed under supremum, *i.e.*

$$h_i \in C \text{ and } \bigwedge h_i \neq 0 \implies \bigvee h_i \in C .$$

• In particular, points belong to all possible connections on $P(E)$ and pulses to all connections on functions $T^E$. Thus they are said to constitute canonic families $S$. 
• **Connected opening**: Let \( C \) be a connection on lattice \( \mathcal{L} \) of canonic family \( S \). For every \( s \in S \), the operation \( \gamma_s : \mathcal{L} \to \mathcal{L} \) defined by

\[
\gamma_s (f) = \bigvee (p \in C , s \leq p \leq f) \quad f \in \mathcal{L} ,
\]

is an *opening*:

- of (point, pulse) marker \( s \)
- and of invariant sets \( \{p \in C, s \leq p\} \cup \{0\} \).

Moreover, when \( r \leq s \), with \( r,s \in S \), then \( \gamma_r \geq \gamma_s \).

• *N.B.* Operation \( \gamma_s \) belongs to the class of the so called *openings by reconstruction*, where each connected component is either suppress or left unchanged. However, such openings can also be based on criteria other than set markers (*e.g.* area, diameter).
Conversely, the $\gamma_s$’s induced by connection $C$ do **characterise** it:

- **Induced Connection**: let $C$ be a sup-generating family in lattice $\mathcal{L}$. Class $C$ defines a connection iff it coincides with invariant sets of a family $\{\gamma_s, s \in S\}$ of openings such that
  
  (iv) for all $s \in S$, we have $\gamma_s(s) = s$,

  (v) for all $f \in \mathcal{L}$, and all $r, s \in S$, the openings $\gamma_r(f)$ and $\gamma_s(f)$ are either identical or disjoint, i.e.

  \[\gamma_r(f) \land \gamma_s(f) \neq 0 \Rightarrow \gamma_r(f) = \gamma_s(f),\]

  (vi) for all $f \in \mathcal{L}$, and all $s \in S$, $s \not\equiv f \Rightarrow \gamma_s(f) = 0$

- **Optimal Segmentation**: the family of the maximal connected components $\leq f$, $f \in \mathcal{L}$, partitions $f$ into elements defined by $\gamma_s(f)$, and one cannot segment $f$ with **less** elements of $C$. 
Properties of the Connections

• **Lattice of the Connections**: The set of the connections that contain the canonic sup-generating class $S$ forms a complete lattice where

$$\inf \{C_i\} = \bigcap C_i \quad \text{et} \quad \sup\{C_i\} = C\bigcup C_i$$

• **Connected Dilations**: Let $C$ be a connection and $S \subseteq C$ a sup-generating class. If an extensive dilation $\delta$ preserves connection on $S$, it preserves it also on $C$.

  – Ex: in $\mathcal{P}(E)$, if the (extensive) dilates of the points are connected, that of any connected component is connected too.

• **Corollary**: The erosion and the opening adjoint to $\delta$ treat the connected components of any $a \in \mathcal{L}$ independently of each other.
Firstly, the erosion $X \ominus B_\lambda$ suppresses the connected components of $X$ that cannot contain a disc of radius $\lambda$; then the opening $\gamma^{\text{rec}}(X ; Y)$ of marker $Y = X \ominus B_\lambda$ «re-builds» all the others.

Application: Filtering by Erosion-Reconstruction

- **Initial image**
- **Eroded of a) by a disc**
- **Reconstruction of b) inside a)**
Comment: efficient algorithm, except for the particles that hit the edges of the field.
**Definition:**

- An operator \( \psi : \mathcal{L} \to \mathcal{L} \) is said to be **connected** when its restriction to \( \mathcal{D} \) is extensive. The most useful of such operations are those which, in addition, are **increasing** for \( \text{TE} \).

**Properties when \( \phi = 0 \):**

- All **binary** reconstruction increasing operations induce on \( \mathcal{L} \), via the cross sections, increasing connected operators on \( \mathcal{L} \).
- The properties to be strong filters, to constitute semi-groups, etc. are also transmitted to the connected operators induced on \( \mathcal{L} \).
- Note that a mapping may be anti-extensive on \( \mathcal{L}^E \), and extensive on \( \mathcal{D} \) (e.g. reconstruction openings). However, the reconstruction closings on \( \mathcal{L}^E \) are also closings on \( \mathcal{L} \).
An Example of a Pyramid of Connected A.S.F.'s

Flat zones connectivity, (i.e. $\varphi = 0$).
Each contour is preserved or suppressed, but never deformed: the initial partition increases under the successive filterings, which are a strong semi-group.
Second Generation Connection

We will now use a dilation $\delta$ to create a new connections $C'$ from a first one $C$ (of associated opening $\gamma_x$).

- **Inverse Images**: Let $\delta : \mathcal{L} \to \mathcal{L}$ be an extensive dilation that preserves connection $C$ (i.e. $\delta(C) \subseteq C$). Then, the inverse image $C' = \delta^{-1}(C)$ of $C$ is still a connection on $\mathcal{L}$, which is richer than $C$, i.e. $C' \supseteq C$.

- **Connected Opening**: If, in addition, $\mathcal{L}$ is infinitely $\lor$-distributive, then the $C$-components of $\delta(a)$ are exactly the images of the $C'$-components of $a$. The opening $\nu_x$ of $C'$ is given by

$$\nu_x(a) = \gamma_x \delta(a) \land a \quad \text{when } x \leq a ;$$
$$\nu_x(a) = 0 \quad \text{when not} .$$
Comment: One want to find the particles from more than 20 pixels apart. They are the only connected components to be identical in both $C$ and $C'$ connections, i.e. the particles whose dilates of size 10 miss the SKIZ of the initial image.
Goal: Extract the osteocytes present in a sequence of 60 sections from confocal microscopy

- Photographs a) and b): sections 15 and 35 respectively;
- Image c): supremum $M$ of the 60 sections.
Application: 3-D Objects Extraction (II)

- **d)**: Threshold **c)** at level 60; **e)**: Connected opening of **d)**

- **f)**: Infinite geodesic dilation of the thresholded sequence (level 200) inside mask **e)** - perspective display -
Another example: Connections in a Time Sequence

Part of the sequence

Connections obtained by cube dilation of size 3 in Space⊗Time (in grey, the clusters)

Representation of the ping-pong ball in Space⊗Time
• **Definition:** E is a (discrete or continuous) metric space. Choose a positive function $\phi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be which is continuous at the origin. A function $g : E \rightarrow T$ is said to be equicontinuous of module $\phi$ when

$$|g(x) - g(y)| \leq \phi [ d(x,y)] \quad (d = \text{distance in } E)$$

The class of these functions is denoted by $G_\phi$

• **$G_\phi$ Lattices:** For each $\phi$, $G_\phi$ turns out to be a totally distributive sub-lattice of $T^E$. All its elements are finite, except possibly its two extrema.

• **Convergence:** In each $G_\phi$ the convergences in Matheron sense and Hausdorff sense (when $E$ is compact) coincide with the pointwise convergence, which, in addition, is uniform.

  (i.e. « $g_n \rightarrow g$ as $n \rightarrow \infty$ » just means « $g_n(x) \rightarrow g(x)$, $x \in E$ »).
Examples of Modules

• **Constant** Functions :
  \[ \varphi = 0 ; \]

• Functions with a *bounded variation* \( k \) :
  \[ \forall d : \varphi (d) \leq k \]

• **Lipschitz** Functions :
  \[ \varphi (d) = k \cdot d \]

• **Geodesic Lipschitz** Functions :
  \[ d \leq d_0 \Rightarrow \varphi (d) = k \cdot d \]
Properties of Equicontinuous Classes

• *Every* $G_\varphi$:
  - contains all constant functions;
  - is self-dual ($g \in G_\varphi \iff -g \in G_\varphi$);
  - is closed under addition by any constant.

• *Dilations:* $G_\varphi$ is closed under the usual dilations and erosions (Minkowski, geodesic), and all these operations are continuous;

• *Filters:* hence $G_\varphi$ is also closed under all derived filters (openings, closings, ASF, etc..), which turn out to be continuous operations;

• *Continuity* is enlarged into module preservation, a stronger notion, which is valid for both continuous and digital cases.
**Weighted Sets**

- **Definition:** Given a module $\varphi$, with each pair $(A, g)$ of the product space $\mathcal{P}(E) \times G_\varphi$ associate the restriction $g_A$ of $g \in G_\varphi$ to $A$, *i.e.* the function

$$
\begin{align*}
g_A(u) &= g(u) & \text{if } u \in A \\
g_A(u) &= 0 & \text{if } u \notin A.
\end{align*}
$$

By so doing, we replace the indicator function of set $A$ by a (variable) weight $g$ which belongs to $G_\varphi$. Hence $g_A$ turns out to be a **weighted set**. As the pair $(A, g)$ spans $\mathcal{P}(E) \times G_\varphi$, the $g_A$’s generate the set $\mathcal{P}_\varphi(E)$.

- **Lattice of the Weighted Sets:** Set $\mathcal{P}_\varphi(E)$ is a complete lattice for the usual ordering $\leq$; in this lattice,

  - the supremum $\bigcup (g_A)_i$ of a family $\{(g_A)_i, i \in I\}$ is the smaller element of $G_\varphi$ which is larger than $\bigvee (g_A)_i$ on $\bigcup A_i$.
  - the infimum, simpler, is given by $\bigcap (g_A)_i = (\wedge g_i) \cap A_i$.
Examples of Weighted Sets

- **First example**: for $\varphi = 0$; the two sets are flat, but with different heights:
  - their $\varphi$-sup is their flat envelope (continuous lines),
  - their $\varphi$-inf is just the intersection of the two functions (dark zone)

- **Second example**: $\varphi$ is a straight line:
Weighted Partitions

The weighted approach extends directly to partitions.

• **Definition**: A **weighted partition** \( x \rightarrow (g_D)_x \) is a mapping \( E \rightarrow P_\phi(E) \) such that
  
  (i) \( \forall x \in E, \quad x \in D(x) \)
  
  (ii) \( \forall (x, y) \in E, \text{ either } (g_D)_x = (g_D)_y \text{ or } (g_D)_x \land (g_D)_y = 0 \)

• **Sub-mappings**: Clearly, the sub-mappings
  
  - \( x \rightarrow D(x) \) is a usual partition, *i.e.* \( D \in D \)
  
  - \( x \rightarrow f(x) = \lor \{(g_D)_y, y \in E\}(x) \) is a usual function of \( T^E \), so that a weighted partition may be denoted by \( \Delta = (D, f) \).

• **Function Representation**: Every function \( f : E \rightarrow T \) can be represented, in different ways, as a \( \lor \{(g_D)_x, x \in E\} \). It suffices to partition \( f \) into zones on which \( f \) admits module \( \phi \) (for example, on which \( f \) is constant).
Theorem (J.Serra): Denote by $\mathcal{L}$ the set of the weighted partitions. Then, the relation

$$\Delta \preceq \Delta' \iff \{ D \preceq D' \text{ in } \mathcal{D}, \text{ and } f \preceq f' \text{ in } T^E \}$$

defines an ordering on $\mathcal{L}$ to which is associated a complete lattice.

Sup and Inf: In $\mathcal{L}$, the supremum $\bigvee \Delta_i$ of family $\{ \Delta_i \}$ admits $D = \bigvee D_i$ for partition. Each class $D(x)$ of $D$, has for weight $g$ the smaller $\varphi$-continuous function larger than $\bigvee (g_D)_i$ on $D(x)$. The $\mathcal{L}$ infimum $\bigwedge \Delta_i$ is given, at each point $x$, by $\bigwedge g_{D_i(x)}$ restricted to $\bigcap D_i(x)$.

Extrema: $\Delta_{\text{max}}$ is the single class partition, weighted by $m$, and $\Delta_{\text{min}}$ is the partition into all points of $E$, each of them being weighted by 0.
An Example of Flat Weighted Partition

- **Partitions**: for $\varphi = 0$, given function $f$:
  - when $f(x) \neq 0$, every subset of the flat zone of $f$ that contains point $x$ can serve as a $D(x)$, with weight $f(x)$;
  - when $f(x) = 0$, class $D(x)$ is reduced to $\{x\}$.

  *(Note that $f$ admits a largest flat partition $\Delta$)*

- **Ordering**: the two largest flat partition $\Delta$ and $\Delta'$ generated from the flat zones of $f$ and $f'$ are not comparable in $\mathcal{L}$, although $f > f'$ (but in $\mathcal{TE}$ !)

Their inf $\Delta \land \Delta'$ is given by two flat sub-zones of $f'$ and 0 elsewhere.

*Functions $f$ and $f'$

*Projection of their infimum partition $\Delta \land \Delta'$*
Comment: Here the weights are taken constant in each flat zone of $f$ and $f'$, i.e. $\varphi = 0$. This generates two weighted partitions $\Delta$ and $\Delta'$. 

- **a**) Non comparable weighted partitions $\Delta$ and $\Delta'$

- **b**) Function associated with supremum $\Delta \vee \Delta'$

- **c**) Function associated with infimum $\Delta \wedge \Delta'$
Cylinders in $\mathcal{L}$

- **Cylinders**: With any weighted set $g_A \in \mathcal{P}_\phi(E)$, it is always possible to associate a weighted partition $\Delta_A$ as follows:

  $x \rightarrow g_A$ if $x \in A$

  $x \rightarrow \{x\}$ if $x \notin A$.

  $\Delta_A$ is composed of class $g_A$ plus a jumble of points, all being weighted by 0. Such a partition is called a cylinder, in $\mathcal{L}$, of base $A$.

- **Sup-generators**: Every weighted partition $\Delta$ turns out to be the $\vee$ of all cylinders $\Delta_{Dx}$ associated with each class $(g_D)_x$ of $\Delta$. Hence the class of the cylinders is sup-generating.

- **Closure under $\vee$**: The supremum $\Delta_A = \vee \Delta_{Ai}$ of family $\{\Delta_{Ai}\}$ of cylinders has for partition classes $\{\cup A_i\}$, plus all $\{x\} \subseteq [\cup A_i]^c$. Hence $\Delta_A$ is itself a cylinder.
Connections on Weighted Partitions

Suppose now that $E$ is equipped with a connection $C_0$. If the bases $C_i$’s of cylinders $\Delta_{C_i}$ are connected and if $\cap C_i \neq \emptyset$, then $\gamma \Delta_{C_i}$ is a cylinder with a connected basis. Now, such cylinders are still sup-generating. Hence,

- **Connection on $\mathcal{L}$**: the cylinders $\Delta_C$ with a connected basis $C$ in $E$, generate a connection $C$ over $\mathcal{L}$.

- **Associated opening**: Given a weighted partition $\Delta = (D, f)$, the point opening $\gamma_x(\Delta)$ of connection $C$ extracts the cylinder whose base is the class $D(x)$ of $D$ covering point $x$, and weight the values of $f$ inside $D(x)$.

In $\mathcal{L}$, the connected opening at point $x$ is a cylinder.
Typology for the Connections on Functions

Module $\phi$

1) $\phi = 0$
   - Constant functions
   - Flat zones

2) $\phi (d) \leq k$
   - Functions whose range of variation $= k$

3) $d \leq d_0 \Rightarrow \phi (d) = k \cdot d^\alpha$
   - Lipschitz geodesic functions
   - Zones in which the variation of $f$ is $\leq k$, and jumps from one zone to another
   - Zones in which the variation of $f$ is smooth, but not from one zone to another
An Example of Jump Connection in $\mathcal{L}$

a) Initial image: gaz burner

b) Jump of size 12: 783 tiles

c) Jump of size 24: 63 tiles

d) Number of tiles versus jump values
Other Example of Jump Connection in $\mathcal{L}$

a) Initial image: polished section of alumine grains

b) Jump connection of size 12:
   - in dark, the point connected components
   - in white, each particle is the base of a cylinder

c) Sketch of the set of the dark points of image b)
Comment: the two phases of the micrograph cannot be distinguished by means of jump connections.

a) Initial image: rock electron micrograph

b) Jump connection of size 15.

c) Jump connection of size 25.
An Example of Smooth Connection in $\mathcal{L}$ (II)

Comment: The smooth connection differentiates correctly the two phases according to their roughnesses.

a) Initial image: rock electron micrograph.

b) Filtering of Image which yields a correct segmentation of a).

c) smooth connection of slope 6 (in dark, union of all point connected components).
Methodology: A jump connection of range 14 for the luminance yields 94 zones. The three color channels are averaged in each of the 94 regions.
References (I)

On binary Connections:

On Connections for Numerical Functions:
- J. Serra Connections for sets and Functions (to appear in Fundamenta Informaticae).

On examples:
On Connected Operators:

- **J. Serra, Ph. Salembier** Connected operators and pyramids. *In SPIE, Vol. 2030, morphological image processing*, San Diego, July 1993, pp. 65-76.