Chapter IV: Morphological Filtering

Serial composition
- \( \theta \psi, \theta \psi \theta \)
- Alternating sequential filters

Parallel composition
- Morphological centers
- Contrast enhancement
In signal processing, the word "filters" is somewhat fuzzy and its meaning is context dependent. It may have the connotation of a convolution, or may denote any system which transforms one image into another. In mathematical morphology, the word has the following precise meaning:

**Definition (G. Matheron, J. Serra):**
A *morphological filter* is an increasing and idempotent transformation of a complete lattice into itself.

**Increasingness**

- This requirement is the most fundamental. It ensures the preservation of the order relation, after filtering.
- This property implies some information loss during the filtering process.
Idempotence

- By definition, an **idempotent operator** transforms any original signal, or image, into an invariant one for the said operator (Ch. I).

- This property appears often, but in an implicit way, in descriptions. An optical filter will be qualified of «red», an amplifier of «band-pass», for example. In Morphological filtering, the property becomes explicit, and compulsory.

- Idempotence may be reached after a single pass, or as a limit under iteration. It may concern a unique operation, or a whole sequence.

- Finally, note that when linear filters are idempotent, then they do not admit inverses: they lose information, which make them a bit more «morphological».
How to obtain Morphological Filters?

Up to now, only two types of morphological filters have been seen: opening and closing.

To create new other ones, two modes of combination may be used:

**Serially**: Composition products of basic filters. It results in the alternating (sequential) filters.

**Parallel**: Combination of basic filters with sup. and inf. It results in the morphological centre.
Alternating Filters (I)

Let $\zeta$ and $\psi$ be two ordered filters (with $\zeta \geq \psi$). By combining them, one can generate four filters which satisfy the following properties:

- **Theorem (G. Matheron):**

1) $\zeta \psi, \psi \zeta, \zeta \psi \zeta, \psi \zeta \psi$ are morphological filters (i.e. increasing and idempotent mappings);

2) $\psi \leq \psi \zeta \psi \leq \zeta \psi \zeta \leq \zeta$

3) $\zeta \psi \zeta$ is the smaller filter larger than $\zeta \psi \vee \psi \zeta$

4) $\psi \zeta \psi$ is the larger filter smaller than $\zeta \psi \wedge \psi \zeta$

We have the following equivalences $\zeta \psi \zeta = \psi \zeta \Leftrightarrow \psi \zeta \psi = \zeta \psi \Leftrightarrow \psi \zeta \geq \zeta \psi$

*N.B.:* 1-The composition of more than three operators does not give another filter. Since, as a filter, $\zeta \psi$ is idempotent, hence $\zeta \psi \zeta \psi = \zeta \psi$.

2- the result extends to all ordered pairs of under-filters $\zeta$ (i.e. $\zeta \geq \zeta \zeta$), and over-filter $\psi$ (i.e. $\psi \psi \geq \psi$) (H. Heijmans)
Alternating Filters (II)

- Proof of the theorem:

1) \[ \zeta \geq \zeta \zeta \geq \zeta \psi \zeta \geq \psi \psi \zeta \geq \psi \zeta \geq \psi \zeta \psi \geq \psi \]

2) idempotence of \( \zeta \psi, \psi \theta \zeta, \psi \zeta \psi, \zeta \psi \zeta \):

\[ \psi \zeta \geq \psi \zeta \zeta \geq \psi \zeta \psi \zeta \geq \psi \psi \psi \zeta \geq \psi \zeta \psi \; \]

\[ \psi \zeta \psi \geq (\psi \zeta \psi \zeta) \psi \geq \psi \zeta \psi \psi \zeta \psi \geq \psi \psi \psi \psi \zeta \psi \geq \psi \zeta \psi \; \]

3) smallest majorant: if filter \( \psi \psi \geq \psi \zeta \zeta \psi \), then

\[ \phi = \phi \phi \geq \zeta \psi \psi \zeta = \zeta \psi \zeta ; \]

4) \[ \psi \zeta = \zeta \psi \zeta \Rightarrow \psi \zeta \geq \psi \psi \psi \zeta \geq \psi \zeta \; \]

\[ \psi \zeta \geq \zeta \psi \Rightarrow \zeta \psi \zeta \geq \psi \psi \zeta \geq \psi \zeta \geq \psi \zeta \zeta \geq \zeta \psi \zeta \; \]
Ordering relations between alternating filters
Comparison between $\gamma \phi$ and $\phi \gamma$

- The more often, the two basic primitives used in the composition are an opening $\gamma$ and a closing $\phi$. Then the previous theorem states that, in general, there is no order relation between $\gamma \phi$ and $\phi \gamma$:

  - There is a "classical" case which illustrates point 4) of the previous theorem, namely that of the filters by reconstruction (Ch VI-13). Then the products $\gamma \phi$ and $\phi \gamma$ are ordered, with $\gamma \phi \geq \phi \gamma$
"Top hat" Extension

**Goal**

The basic "Top hat" highlights peaks but does not remove dense fluctuations, such as noise. If we wish to compensate both smooth signal variations and also noise, then we can start from operator:

\[ \gamma \varphi \wedge I \]

and take its residue

\[ \rho(f) = f - \{ \gamma \varphi(f) \wedge f \} \]

(for a given function f).

In particular, when \( \gamma \varphi \) is a strong filter (e.g. 1-D opening and closing, or reconstruction ones) then \( \gamma \varphi \wedge I \) turns out to be an algebraic opening.
Goal:
The notions of sup-, inf- and strong filters are introduced to describe the properties of composition products such as $\theta \psi$, and their robustness i.e. the sensitivity of the result with respect to variations of the input signal.

Definition:
\begin{align*}
\vee \text{- filter if } & \psi = \psi(I \vee \psi) \\
Filter \ \psi \ \text{is } \wedge \text{- filter if } & \psi = \psi(I \wedge \psi) \\
\text{strong if } & \psi = \psi(I \vee \psi) = \psi(I \wedge \psi)
\end{align*}

Theorem (G. Matheron):
\begin{itemize}
\item A mapping is a $\vee$-filter (resp. $\wedge$-filter) if and only if it can be decomposed into a product $\gamma \varphi$ of an opening with a closing (resp. $\varphi \gamma$).
\item If $\varphi$ is a closing and $\gamma$ an opening, then $\varphi$ and $\gamma$ are strong filters, $\gamma \varphi$ and $\varphi \gamma$ are $\vee$-filters, $\varphi \gamma$ and $\gamma \varphi \gamma$ are $\wedge$-filters.
\end{itemize}
Geometrical Interpretation

The $\lor$-filter property ensures that any signal $g$ between $f$ and $f \lor \psi(f)$ gives the same filtered version as $f$ itself, i.e. $\psi(g) = \psi(f)$. The $\land$-filters yield the dual result. Finally, in case of strong filters, this robustness is extended to all signals which are comprised between $f \land \psi(f)$ and $f \lor \psi(f)$.
The number of distinct filters that can be created by composing two filters is rather restricted. If we want to go further, the composition must involve not two filters but two families of filters. This idea leads to alternating sequential filters.

**Theorem (J.Serra):**
Consider two families \( \{ \zeta_i \} \) of under-filters and \( \{ \psi_i \} \) of over-filters such that:
\[
\{ \zeta_i \} \text{ increases with } i , \quad \{ \psi_i \} \text{ decreases with } i ,
\]
\[
\psi_n \leq \ldots \leq \psi_2 \leq \psi_1 \leq \zeta_1 \leq \zeta_2 \leq \ldots \leq \zeta_n \leq \psi_1 \leq \zeta_1
\]
then the two products of composition
\[
N_i = \zeta_i \psi_i \ldots \zeta_i \psi_2 \zeta_1 \psi_1 \quad \text{ and } \quad M_i = \psi_i \zeta_i \ldots \psi_2 \zeta_2 \psi_1 \zeta_1
\]
are idempotent, and called Alternating Sequential Filters.
Alternating Sequential Filters (II)

• Proof of the theorem

1) Put $\mu_i = \psi_i \zeta_i$. For $j \geq i$, one has $\mu_j \mu_i \leq \mu_j \leq \mu_i \mu_j$, since

$$\psi_i \leq \zeta_i \leq \zeta_j \Rightarrow \psi_i \zeta_i \leq \zeta_i \zeta_i \leq \zeta_i \leq \zeta_j \Rightarrow \psi_j \zeta_j \psi_i \zeta_i \leq \psi_j \zeta_j \zeta_j \leq \psi_j \zeta_j$$

(same proof, *mutatis mutandis*, for the second inequality)

2) $M_i M_i = (\mu_i \mu_{i-1} \ldots \mu_1)(\mu_i \mu_{i-1} \ldots \mu_1) \leq \mu_i (\mu_i \mu_{i-1} \ldots \mu_1) \leq (\mu_i \mu_{i-1} \ldots \mu_1) = M_i$

but also: $M_i M_i = (\mu_i \mu_{i-1} \ldots \mu_1)(\mu_i \mu_{i-1} \ldots \mu_1) \geq \mu_i (\mu_i \mu_{i-1} \ldots \mu_1) = M_i$.

• Absorption Law

$$j \geq i \Rightarrow M_j M_i = (\mu_j \ldots \mu_i+1)(\mu_i \mu_{i-1} \ldots \mu_1)(\mu_i \mu_{i-1} \ldots \mu_1) = M_j$$

( ...but $M_i M_j \leq M_j$ ! )
Properties of alternating sequential filters

Duality

• Theoretically, alternating sequential filters are not self-dual, that is:

\[ M_i(m - f) \neq m - M_i(f) \]

In practice, the difference is often negligible.

Compatibility with magnifications

• If the families of primitives are compatible with scale modifications, that is:

\[ \psi_k(h_k(.)) = h_k(\psi_1(.)), \quad \zeta_k(h_k(.)) = h_k(\zeta_1(.)) \]

where \( h_k(.) \) represents a spatial or temporal similarity, then the resulting alternating sequential filters are also compatible.

• Intuitively, this property indicates that the filter of order \( k \) works at a scale \( k \) as a filter of order 1 at a scale 1.
Usefulness of alternating sequential filters (I)

Comment: In general, the two basic families transformations are granulometries. The F.A.S. are very useful when dealing with noisy signals. The following example shows that a simple $\varphi \gamma$ is not appropriate to cancel noise, whatever the size of the structuring element is.
Usefulness of alternating sequential filters (II)

An alternating sequential filter allows noise cancellation and smoothly produces a good approximation of the signal without noise.
Pyramids

• The absorption law (1), below, of the ASF’s, suggests we focus on families \( \{ \psi_\lambda \} \) of operators that depend on a positive parameter, \( \lambda \) say.

• For generating nice properties, the \( \{ \psi_\lambda \} \) must satisfy some consistency as \( \lambda \) varies. In particular, they constitute a pyramid of operators when each \( \psi_\lambda(f) \) may be obtained from any transform \( \psi_\mu(f) \), for \( 0 < \mu \leq \lambda \), i.e.

\[
\lambda \geq \mu > 0 \implies \text{there exists } \nu > 0 \text{ such that } \psi_\lambda = \psi_\nu \psi_\mu
\]

• The two basic types of pyramids are the following:

\[
\begin{align*}
\lambda \geq \mu > 0 & \implies \psi_\lambda \circ \psi_\mu = \psi_\lambda & (1) \quad \text{(the stronger imposes its law)} \\
\lambda \geq \mu > 0 & \implies \psi_\lambda \circ \psi_\mu = \psi_{\lambda + \mu} & (2) \quad \text{(the effects are additive)}
\end{align*}
\]

• Note that the condition of being granulometric semi-group (see III-15) is more demanding than that of forming a pyramid, since then we have

\[
\lambda, \mu > 0 \implies \psi_\lambda \circ \psi_\mu = \psi_{\sup(\lambda, \mu)}.
\]
An Example of a Pyramid

ψ_\lambda := \text{Hexagonal alternating sequential filter of size } \lambda \text{ beginning by a closing ;}
(a) := \text{Initial image } f ;
(b) := \psi_2(f) ; \quad (c) := \psi_4(f) ; \quad (d) := \psi_7(f) = \psi_7[\psi_4(f)]
(e) := \psi_2[\psi_4(f)] \neq \psi_4(f) \quad (\text{the non identical pixels are in white})
The semi-group $\psi_\lambda \psi_\mu = \psi_\mu \psi_\lambda = \psi_{\text{sup}(\lambda,\mu)}$

- The absorption law $\lambda \geq \mu > 0 \Rightarrow \psi_\lambda \circ \psi_\mu = \psi_\lambda$ (1) implies that:
  - the $\psi_\lambda$ are idempotent (not necessarily increasing), of invariant class $B_\lambda$
such that $\lambda \geq \mu > 0 \Rightarrow \psi_\lambda (B_\mu) = B_\lambda$ (2)
  - however, rel. (2) does not imply relation (1).

For generating a semi-group, the additional law

$$\lambda \geq \mu > 0 \Rightarrow \psi_\mu \circ \psi_\lambda = \psi_\lambda$$

(3)

is equivalent to

$$B_\lambda \subseteq B_\mu$$

(4)

- Rel. (1) has a \textit{markovian} meaning, since it suffice to know level $\mu$ for being able to determine all the upper ones (\textit{i.e.} those $\psi_\lambda$ for which $\lambda \geq \mu$);

- When combined with rel.(3), it becomes similar to \textit{waveband}: $\psi_\lambda$ smoothes “more” than $\psi_\mu$ since it amends the same fine details without introducing new ones.
Methodology

With the objective of creating new filters without compositions, let us consider now a family of primitives and a filter whose output is always one of the primitives values. In a first step, the selection rule relies on sup. and inf. but it can be extended to more complex rules such as those leading to "toggle mapping".

Combination with sup. and inf.

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**Criterion**

\[ \psi_1 \]
\[ \psi_2 \]
\[ \ldots \ldots \ldots \]
\[ \psi_{n-1} \]
\[ \psi_n \]

**Primitives**

- Combination with sup. and inf.
- Complex rules "toggle mapping"

**Examples:**
- Morphological center
- Contrast enhancement
The Morphological Centre

The aim of this transform is to create self-dual filters. It is a kind of gravity centre for lattice structures.

**Definition (J. Serra):**

The morphological centre with respect to the family \( \{\psi_i, i \in J\} \) is defined as:

\[
\beta = [I \lor (\land \{\psi_i\})] \land (\lor \{\psi_i\})
\]

In other words, if at point \( x \)
- all \((\psi_i f)(x)\) are greater than the original, the centre is the smallest of them,
- all \((\psi_i f)(x)\) are smaller than the original, the centre is the largest one.

Otherwise the centre is the original signal.

An example of morphological centre
Properties of the Morphological Centre

Self-duality
• If the family of primitives is self-dual as a whole (each primitive has its dual in the family), then the centre is self-dual.

Strength
• If the primitives are strong filters, the centre is also a strong filter (e.g. reconstruction filters)

Idempotence
• In general, the centre is not idempotent. But with any opening $\gamma$ and closing $\phi$, the idempotent iteration $\alpha$ of the centre between $\gamma\phi\gamma$ and $\phi\gamma\phi$ is a strong filter (and self-dual if the opening and closing are dual for each other)

$$\alpha = \beta^n = [ (I \lor \gamma\phi\gamma) \land \phi\gamma\phi ]^n$$

Moreover, at every point $x$ the convergence of the sequence $(\beta f)^n(x)$ is monotonous. Note the difference with the median operator which is also self-dual but not idempotent, and may oscillate indefinitely.
This transformation is a sort of *anti-centre*. Its goal is not to create a signal which is close to the original one but to produce a highly contrasted signal.

**Definition** Given two anti-extensive and extensive transformations and an original signal \( f \), the result is the closest transformation value of \( f \).

**Property** *(F. Meyer, J. Serra)*
- When the primitives are opening and closing, the resulting contrast is *idempotent* (but not increasing).
**Set Activity Lattice**

- **Definition (J.Serra)**: let $\mathcal{L}'$ be the family of the mappings from $\mathcal{P}(E)$ into itself. Consider two elements $\alpha$ et $\beta$ of $\mathcal{L}'$. If we have

  \[ \alpha \prec \beta \iff I \cap \alpha \supseteq I \cap \beta \quad \text{and} \quad I \cup \alpha \subseteq I \cup \beta, \]

  *i.e.* if $\beta$ modifies more points than $\alpha$ does; then $\beta$ is said to be more active than $\alpha$ and one writes $\alpha \preceq \beta$.

- **Activity** turns out to be an ordering relation on class $\mathcal{L}'$. It induces the so-called **Activity Lattice**, where the sup $\vee$ and the inf $\wedge$ of family $\{\psi_i, i \in I\}$ are given by

  \[ \vee \psi_i = [C \cap (\cup \psi_i)] \cup (\bigcap \psi_i) \quad \text{and} \quad \wedge \psi_i = [I \cap (\cup \psi_i)] \cup (\bigcap \psi_i) \]

- When family $\{\psi_i\}$ is self-dual, *i.e.* when for any $i \in I$ we have $\psi_i = C \psi_j C$ for some $j \in I$, then $\vee \psi_i$ and $\wedge \psi_i$ become self-dual.
Although the activity ordering yields a **complete lattice** in the set case, it is an **inf semi-lattice** only in case of numerical functions.

- Typically, the centre $\beta$ of family $(\psi_i)$ is nothing but the **activity inf** of the $(\psi_i)$.

- **Supremum**: The **activity sup** $\omega$ of mappings $(\psi, \zeta)$ exists if and only if

$$\{ x: \psi_x < I_x \} \cap \{ x: \zeta_x > I_x \} = \emptyset$$

Then it has a value

$$\omega_x = \psi_x \text{ if } \psi_x < I_x ; \quad \omega_x = \zeta_x \text{ if } \zeta_x > I_x ; \quad \omega_x = I_x \text{ if } \psi_x = \zeta_x = I_x$$

**Examples**:  

- Contrast enhancements by **toggle mappings**;
- function **levelling** (Ch. VI).
On theory of Morphological Filtering:
- The theory of morphological filtering is due to G. Matheron {MAT88a}. Two simpler, and illustrated, presentations can be found in {MEY89c} {SER92c} and a systematic one in {RON91}.
- The links with linear filters are studied in {MAR87b}.

On Alternating Filters:
- Alternating sequential filters were proposed by S. R. Sternberg {STE83} and the corresponding theory made by J. Serra {SER88}.
- The top hat extension comes from {SAL91}.

On Activity Lattice:
- Centre and activity lattice appear for the first time in {SER88, ch.8}; the two major derivations of this approach are toggle mappings {SER89} and contrast operators {MEY89b}.