DIGITAL SKELETONS IN EUCLIDEAN AND GEODESIC SPACES

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Abstract

This paper is an attempt to bridge the gap between two classes of skeleton algorithms widely used in Mathematical Morphology: the skeleton by maximal balls and the skeleton by thinnings.

First, we show that the first type of skeleton, usually resulting from morphological openings according to Lantuéjoul's formula, can also be obtained by non homotopic thinnings. Moreover, these thinnings are not sequential but based on the intersection of elementary thinnings.

The second step is a restrictive selection among the previous structuring elements which preserve homotopy. It leads to the definition of a connected skeleton containing the skeleton by maximal balls. This algorithm is given in the 2D Euclidean space for both the hexagonal and square grids. This skeleton combines the advantages of the two classes of skeletons and avoids the main drawbacks involved by rotating thinnings for building connected skeletons.

The extension of this definition to geodesic spaces is discussed in the third part. The geodesic thinnings are introduced to define a connected skeleton containing the skeleton by maximal balls.

Introduction

There currently exist two kinds of morphological tools to build skeletons: the residues of openings on the one hand, and the thinning transformations on the other hand. In fact, the former are directly derived, by means of the Lantuéjoul's formula [7], from the definition of the skeleton of a set in terms of maximal balls [4]. The latter stem from the definition of a "prairie fire" skeleton [3]. The thinning algorithms are a simulation of the fire propagation. Provided that good conditions are fulfilled by the initial set, it can be shown that these two definitions lead to the same skeleton.

Unfortunately, in both cases, the resulting transforms have many drawbacks. First, the residues of openings do not exactly correspond to the intuitive notion of a skeleton, mainly because the transformation is not homotopic. The result of such a skeleton transform is generally not connected. Although it is not proved that the skeleton of a connected set should be connected, the fulfillment of this condition has led many people to try to find new algorithms to obtain connected skeletons [5,8]. On the other hand, the skeletons by thinning produce connected skeletons. Using thinning is quite natural because they are the only morphological operators which can be homotopic. However, these skeletons are not very handy. In particular, they are not unique because sequences of thinnings are needed to obtain homotopic transforms,. However, there are multiple ways to define the order of the thinnings in the sequences, each sequence producing a different skeleton. Moreover, these skeletons are often biased and their relationship to the non connected skeleton is not obvious [9].

The aim of this paper is to briefly describe a methodology in order to bridge the gap between these two kinds of skeletons. We shall see that this approach leads to the definition of a connected skeleton algorithm which produces a final transform close to the maximal ball skeleton. We shall also see that this technique is not linked to the dimension of the working space. This allows to define 3D and geodesic skeletons.

The geodesic skeleton will be introduced and its use in order to produce a connected skeleton containing the maximal ball skeleton will be explained.

All the notions described here are given in the digital case. Moreover, the hexagonal grid is often used in the examples and in the proofs. However, all the available results may be transposed to the square grid and some of the corresponding transforms will be given as often as possible.

1 Maximal ball skeleton and skeletons by thinnings

In this first part, the definitions of the maximal balls skeletons and of the skeletons by thinnings are reviewed. We also describe the morphological transformations used to build them and their main drawbacks.

1.1 Definitions

1.1.1 Maximal ball skeleton

Let X be a digital set included in Z^n. Let $B(x,i)$ be a ball of radius i centered at point x and included in X. This ball is a maximal ball if and only if there exists no other ball $B(y,j) \subset X$ such that $B(x,i) \subset B(y,j)$.

The skeleton S(X) of X is the set of all the centers of the maximal balls included in X.

(a) (b)

Figure 1: skeleton (b) of a set (a) defined with maximal balls

The residue R(X) of X is the set made of those points of X which do not belong to its opening by an elementary ball:

$$R(X) = X/(X)_B$$

where $(X)_B$ is the opening of X by the elementary ball B.

According to Lantuéjoul's formula, it can be shown that the skeleton is also the set of all the residues of the successive erosions of X:

$$S(X) = \bigcup_i (X \ominus iB)/(X \ominus iB)_B,\ i \in N$$

$$S(X) = \bigcup_i R(X \ominus iB)$$

<u>Proof:</u>

Let x be a point belonging to the opened set $(X \ominus iB)_B$. This point belongs to a ball B included in the eroded set $(X \ominus iB)$. A size i dilation of B produces a ball of size i+1 included in X which covers the ball of size i centered in x. This point, therefore, cannot be the center of a maximal ball of size i included in X. The centers of the maximal balls of size i, if any, must belong to the residue of $(X \ominus iB)$.

Conversely, suppose that a point x, although it belongs to the residue, is not the center of a maximal ball of size i. So, it must be included in a ball of size j > i covering the ball B(x,i). But, if we erode this ball of size j by a ball of size i, we obtain a ball of size j-i ≥1 which contains x. Therefore x belongs to the opened set $(X \ominus iB)_B$which is contradictory with the initial hypothesis. Q.E.D.

Although this skeleton has interesting properties (such as the capability of entirely reconstructing the initial set from the set of maximal balls), it does not preserve the connectivity of the set X (Figure 1).

1.1.2 Skeletons by thinnings

Let T = (T_1 ,T_2) be a two-phase structuring element (T_1 and T_2 are two structuring elements with a common origin). A thinning of X by T is defined by [1]:

$$X \bigcirc T = X / (X \star T)$$

where:

$$X \star T = (X \ominus \check{T}_1) \cap (X^c \ominus \check{T}_2)$$

is the hit-or-miss transform.

Let **T** = {T^1,T^2,...,T^n} be a family of structuring elements. Each T^i is a two-phase structuring element (T^i_1 ,T^i_2). The thinning of a set X by the family **T** is defined by:

$$X \bigcirc \mathbf{T} = X / (X \star \mathbf{T})$$

where:

$$X \star \mathbf{T} = \bigcup_i (X \star T^i).$$

This definition produces a thinning which does not depend on the order in which the various T^i are used.

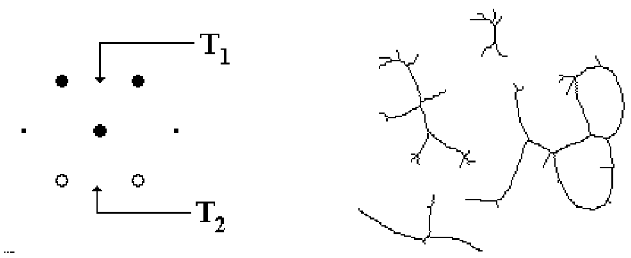
It is not however this kind of thinning which is ordinarily used for building connected skeletons, but sequential thinnings [5]. These thinnings use homotopic structuring elements, that is structuring elements which do not break the connectivity of the initial set. Furthermore, these structuring elements must be used sequentially because, even when each of them preserves homotopy, this preserving quality is not guaranteed when they are used in union thinnings.

By applying successive thinnings with homotopic elements to a set X:

$$(((X \bigcirc T_a) \bigcirc T_b) \bigcirc ...)$$

where T_a, T_b, are structuring elements which preserve homotopy, one can produce homotopic transforms called "connected skeletons".

Unfortunately, there is no rule for choosing the order of the structuring elements and each sequence leads to a different kind of "skeleton". The most classical sequence used on the hexagonal grid is the sequence (L^1, L^2, ..., L^6) where the L^i are the successive rotations of the L structuring element (Figure 2a).



(a) **(b)**

Figure 2: The L structuring element (a) and the corresponding connected skeleton (b)

Although these L-skeletons are indeed connected (Figure 2b), they are not unique. The result of the transformation strongly depends upon the order of the structuring elements in the sequence. The quality of the result also leaves much to be desired. These transformations are called skeletons only because the final result looks like a skeleton. However, there is no obvious link between this transform and the maximal ball skeleton.

1.2 Expressing the maximal ball skeleton with thinnings

1.2.1 General methodology

To bridge the gap between the two types of skeletons, the following methodology is applied:

- The first difficulty comes from the fact that two different morphological transformations are used to define these two types of skeletons. So, to be able to compare the two approaches, we must express the maximal ball skeleton in terms of thinnings. We shall see that the Lantuéjoul's formula can be written as an union of thinnings. These thinnings will use a simple family of structuring elements (without order).
- Next, we shall try to build a connected skeleton by sorting these structuring elements to eliminate those which may break the connectivity. The resulting transform will produce a connected skeleton obtained by union thinnings which guarantee the uniqueness of the result.

1.2.2 Maximal ball skeleton by thinnings

Starting from the Lantuéjoul's digital formula:

$$S(X) = \bigcup_i [(X \ominus iB)/(X \ominus iB)_B]$$

let us define:

$$S_n(X) = \bigcup_{i=0}^{n} [(X \ominus iB)/(X \ominus iB)_B]$$

and prove that the following iterative transform:

$$Z_n = (Z_{n-1} \ominus B) \cup R(Z_{n-1})$$

with $Z_0 = X$, is equal to:

$$Z_n = (X \ominus nB) \cup S_{n-1}(X)$$

<u>Proof</u>:

We first show that Z_n can be written:

$$Z_n = (X \ominus nB) \cup \left[\bigcup_{i=0}^{n-1} (X \ominus iB)/(X \ominus iB)_B \right]$$

Assume that this expression is true at step n-1:

$$Z_{n-1} = (X \ominus (n-1)B) \cup \left[\bigcup_{i=0}^{n-2} (X \ominus iB)/(X \ominus iB)_B \right]$$

To calculate $Z_{n-1} \ominus B$, we apply the following lemma:

<u>Lemma:</u>

Let X and Y be two disjoint sets. If $X \oplus B$ and Y are also disjoint, we have:

$$(X \cup Y) \ominus B = (X \ominus B) \cup (Y \ominus B)$$

It is easy to see that any pair taken from the following sets:

$$X \ominus (n-1)B$$
$$(X \ominus iB)/(X \ominus iB)_B,\ \forall\, i \in [0, n-2]$$
$$(X \ominus jB)/(X \ominus jB)_B,\ \forall\, j \in [0, n-2],\ j \neq i$$

satisfies the conditions of the lemma.

Then:

$$Z_{n-1} \ominus B = (X \ominus nB) \cup \left[\bigcup_{i=0}^{n-2} [(X \ominus (i+1)B) \cap ((X \ominus (i+1)B)^c \ominus 2B)] \right]$$

is equal to $X \ominus nB$.

Similarly:

$$Z_{n-1}/(Z_{n-1})_B = Z_{n-1} \cap [X \ominus (n-1)B]^c_B$$
$$Z_{n-1}/(Z_{n-1})_B = \bigcup_{i=0}^{n-1} [(X \ominus iB)/(X \ominus iB)_B]$$

Finally, we have:
$$Z_n = (X \ominus nB) \cup S_{n-1}(X) \qquad \text{Q.E.D.}$$

At the end of this iterative transform, we obtain the maximal ball skeleton:
$$Z_\infty = \lim_{n\to\infty} Z_n = S(X)$$

But this transformation can also be expressed in terms of thinnings. If we write the general expression $(Z \ominus B) \cup R(Z)$ under the form:
$$(Z \ominus B) \cup (Z \cap (Z)^c_B) = Z \cap [(Z \ominus B) \cup (Z_B)^c] = Z \cap [(Z \ominus B)^c \cap (Z_B)^c]^c$$
We can show that $(Z \ominus B)^c \cap Z_B$ is a hit-or-miss transformation.

Proof:

We have:
$$Z_B = \bigcup_{a \in B} (Z \ominus B_a)$$
and:
$$(Z \ominus B)^c = \bigcup_{b \in B} (Z^c \ominus L_b)$$
where B_a is the ball B translated in the direction a and L_b the structuring element corresponding to the translation of a point in the direction b (a and b are all the possible direction vectors centered at the origin which can be defined in the elementary ball B).

Then, we have:
$$(Z \ominus B)^c \cap Z_B = \bigcup_{a \in B, b \in B} [(Z \ominus B_a) \cap (Z^c \ominus L_b)]$$
$$(Z \ominus B)^c \cap Z_B = \bigcup_{a,b} (Z \bigstar T_{a,b}) = Z \bigstar \mathbf{T}$$
This iterative transformation is a thinning by a family T of structuring elements:
$$T = \{T_{a,b} = (B_a, L_b), \forall\, a,b \in B\} \qquad \text{Q.E.D.}$$

Therefore the maximal ball skeleton can be written:
$$S(X) = (X \circ \mathbf{T}) \circ \ldots \circ \mathbf{T} \circ \ldots$$
It is produced by successive iterations of union thinnings.

In the previous proof, no assumption was made as to the dimension of the working space and as to the sampli6ng grid. This formula holds for the 2D hexagonal or square skeletons as well as for the 3D ones.

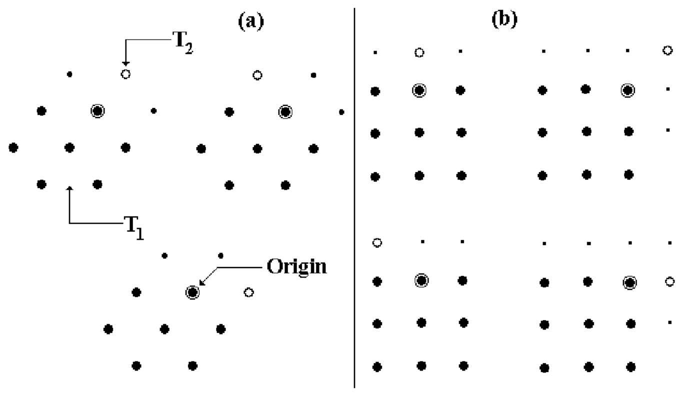


Figure 3: The **T** family of thinnings used for skeletons on hexagonal (a) and square (b) grids

Let us emphasize the **T** family for the 2D skeleton on the hexagonal grid. **T** is made of the following structuring elements (Figure 3a) with all their possible rotations. Notice the position of the origin of these structuring elements. Notice also that these structuring elements are not defined on the elementary hexagon as was the case for the L-structuring element used in the sequences of thinnings. In the same way, the corresponding family can be defined (Figure 3b) for the square grid.

2 From the non-connected to the connected skeleton

To build a connected skeleton from the previous thinnings, two methods are available. The first one consists in analyzing the various structuring elements of the **T** family in order to keep only those which preserve homotopy. However, this procedure may be tedious especially when dealing with the 3D space. The second method is based on the analysis of what is to be done to connect the maximal balls skeleton. The first approach will be explained in the case of the 2D hexagonal skeleton. The general method will be used especially for finding the family of structuring elements for the 3D skeleton. We shall see that, despite the fact that this skeleton is connected and very close to the maximal balls skeleton, some local variations may exist.

2.1 Sorting the configurations preserving homotopy

A thinning configuration belonging to the family **T** is said to preserve the homotopy if the points which are eliminated by the thinning do not modify the connectivity relationships of the set X. If X is connected, it remains connected and the neighborhood relationships between its different connected components are preserved [9].

The checking procedure is the following: for each origin point of the structuring element, we verify that suppressing this point does not change the connectivity. This checking must be performed hierarchically because, as we work at the same time with all the possible structuring elements with all their rotations, the adjacent points may also be eliminated and therefore the neighborhood relationships of the center point can be dramatically modified.

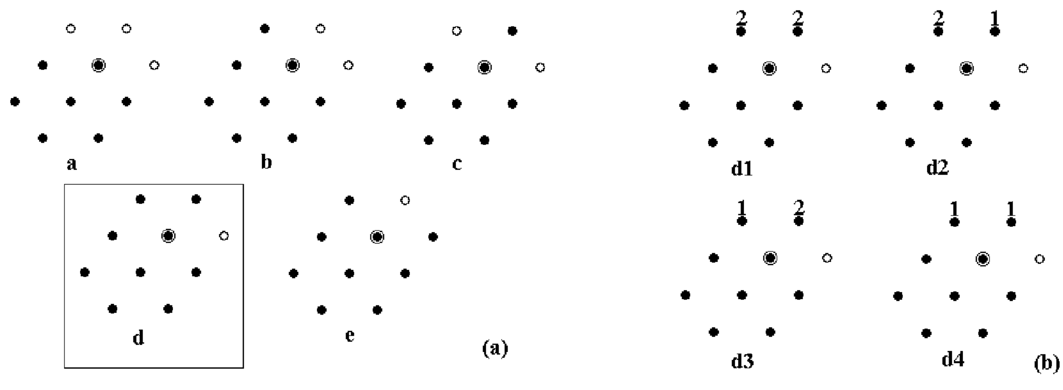


Figure 4: Details of the family T (a) and decomposition of the configuration d (b)

The structuring elements belonging to the family **T** for a given direction are shown in Figure 4a (up to a symmetry). Only the configurations (a) and (c) can be analyzed at this first level. The configuration (a) clearly preserves the homotopy whatever the modifications occuring in the neighborhood of the origin point. On the contrary, the configuration (c) is not homotopic: if the center point is suppressed, the connection is broken.

Let us analyze the configuration (d). As mentioned above, the behavior of the points surrounding the origin is of primary importance. But this behavior strongly depends on their belonging to the opening of X. Any adjacent point belonging to X_B will always be preserved. So, a deeper analysis will split the configuration (d) into four configurations where the belonging of the adjacent points is taken into account. In Figure 4b, the points belonging to X^c are denoted 0, those included in the opened set X_B by 1 and those belonging to the residue R(X) by 2.

It can be shown that the configurations (d1) and (d2) are not homotopic and that the two configurations (d3) and (d4) are equivalent to the configuration (e).

The same decomposition is performed for the configurations (b) and (e). This leads to the final result:

Among all the possible configurations belonging to the family **T**, the only ones that preserve the homotopy and therefore that produce a connected skeleton by union thinning are given (with all their rotations) in Figure 5. The points denoted $\bar{2}$ are points which do not belong to the residues of the set to be thinned.

Figure 5b gives an example of such a connected skeleton.

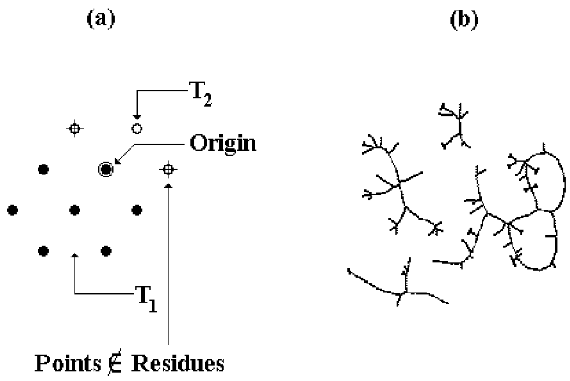


Figure 5: Configurations preserving homotopy (a) and corresponding skeleton (b)

2.2 The general connection scheme

The previous approach, however, is very tedious and may be risky because some configurations may be forgotten. For that reason, another procedure can be used. It is based on the analysis of the connections which must be set to connect, at each step i, those components of the set Z_i (the result of the previous thinning) which belong to the skeleton to the eroded set $Z_i \ominus B$.

Two kinds of connections must be made:

- the connection of the residues obtained at step i with the eroded set of size i+1 (connections of the first kind).
- the connection of the connected components of the eroded set of size i+1, when the disconnection occurs between the erosions of sizes i and i+1. These residues actually correspond to the centers of elementary balls which touch each other (Figure 6a).

The first kind of connections can be obtained by adding to the residues at step i the points which connect these residues and the eroded set after step i. These points are the points of Z_i which are adjacent to the residue $R(Z_i)$. These points, in the case of the hexagonal grid, belong to the set $(R(Z_i) \oplus B) \cap Z_i$. Doing so, we define a new iterative transform:

$$Z_{i+1} = (Z_i \ominus B) \cup [(R(Z_i) \oplus B) \cap Z_i]$$

with $Z_0 = X$.

This can be written under the form of a thinning:

$$Z_{i+1} = Z_i \cap A^c$$

with the hit-or-miss transform A given by:

$$A = (Z_i)_B \cap (Z_i \ominus B)^c \cap (R^c(Z_i) \ominus B)$$

The interpretation of the above equation is the following: Z_{i+1} can be obtained by thinnings using the **T** family of structuring elements (first two terms of A) provided that there is no residue $R(Z_i)$ adjacent to the origin of the structuring element (third term of A).

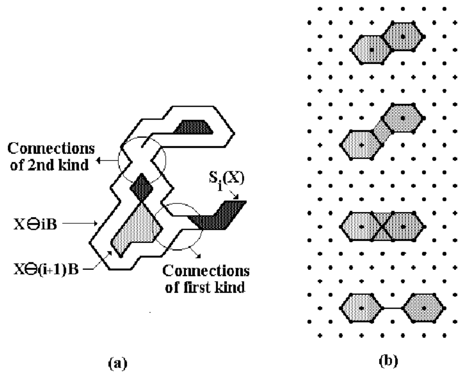


Figure 6: The two connections in the connected skeleton (a) and 2nd kind connections on the hex. grid (b)

Starting from this general scheme, we compute again the structuring elements which preserve the connectivity in the case of a hexagonal grid when B is an elementary hexagon. For preserving the connections of the first kind, these structuring elements are given in Figure 7a (up to their rotations and symmetries). The significance of the coded numbers is given above. This family is denoted **T'**.

For preserving the second kind of connection, we simply have to check all the possible configurations for two elementary balls to touch each other and suppress those which may break the connections. Figure 6b displays all these configurations in the case of the hexagonal grid. The structuring elements which preserve this second kind of connections are given at Figure 7b. The corresponding family is denoted **T''**.

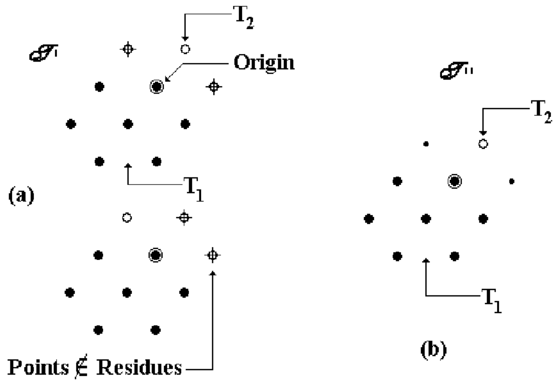


Figure 7: Sub-families T' (a) and T'' (b)

Finally, the family **S** of the structuring elements which preserve both connections is given by:

$$\mathbf{2S} = \mathbf{T'} \cap \mathbf{T''}$$

This set **S** is the same as the family obtained by the direct approach (cf. Figure 5a).

2.3 Application to 2D and 3D skeletons

This approach is particularly helpful for finding the structuring elements family **S** for the 2D skeleton on a square grid and for the 3D connected skeleton.

When we work on a 8-connectivity square grid, the same analysis leads to the structuring elements given Figure 8. Note that the transformation used to achieve the connections of the first kind in the case of the 8-connected square grid is different for the side configurations (a linear dilation is used) and for the corner configurations (a square dilation is needed).

Let us define the corresponding structuring elements in the case of a digital 3D space. The elementary ball is a cuboctahedron and the digital grid is cubic. The second kind of connections correspond to the cuboctahedrons which touch each other (Figure 9a).

The first kind of connections are preserved by checking that no residue point is adjacent to the origin. Finally, it can be shown (as for the 2D hexagonal case) that the configuration given in Figure 9b and all its rotations produce, by union thinnings, a 3D connected skeleton.

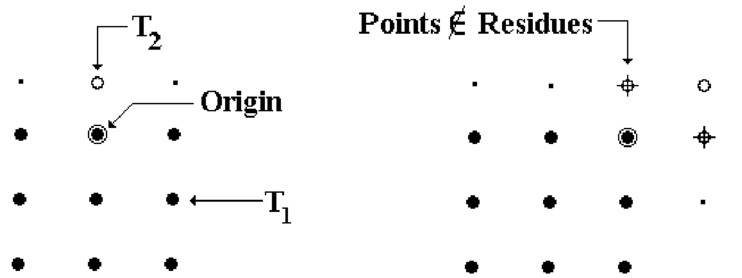


Figure 8: Structuring element for the 8-connected skeleton on the square grid

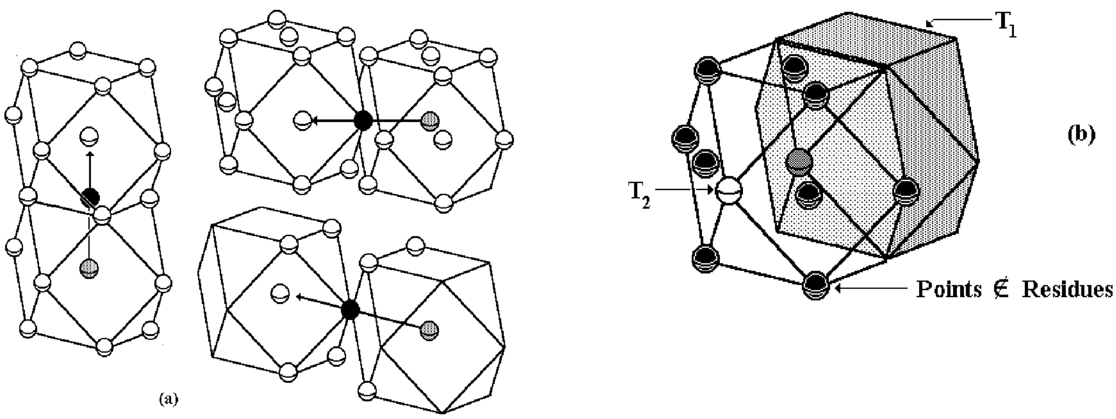


Figure 9: 2nd kind of connections in the cubic space (a) and struc. elements for the 3D connected skeleton (b)

2.4 Correspondence between the maximal balls skeleton and the connected skeleton

A crucial question remains at the end of this procedure: does there exist any relationship between the connected skeleton $S_c(X)$ and the maximal ball skeleton $S(X)$. $S_c(X)$ is supposed to be a connected version of the maximal ball skeleton and therefore it assumed to contain the latter skeleton. In fact, it is not true as is illustrated in Figure 10. This means that, despite the fact that the structuring elements family used for building the connected skeleton is the best possible (adding any other structuring element to this minimum set will produce a non connected skeleton), the final result does not contain the maximal ball skeleton.

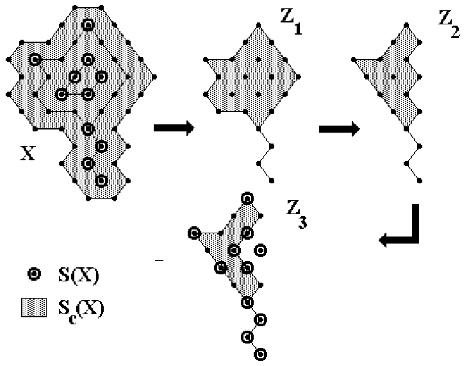


Figure 10: The connected skeleton does not contain the centers of maximal balls

However, the two skeletons are very similar and it is possible to prove that the dilated connected skeleton $S_c \oplus B$ contains the maximal ball skeleton S:

$$\forall X, S_c(X) \oplus B \supset S(X)$$

Proof:

The proof will be given for the 2D hexagonal grid, but it is obviously general.

We know that the connected skeleton is obtained partly by iteration of Z_i:

$$Z_i = (Z_{i-1} \ominus B) \cup [(R(Z_{i-1}) \oplus B) \cap Z_{i-1}]$$

and partly by adding to the set Z_i some points for achieving the connections of the second kind. So:

$$S_c(X) \supset \lim_{i \to \infty} Z_i$$

We also know that Z_i is obtained from Z_{i-1} by union thinnings with the structuring elements family **T'** (see Figure 7a).

We write:

$$Z_i = (Z_{i-1} \ominus B) \cup V_{i-1}, \text{ with } V_{i-1} = (R(Z_i) \oplus B) \cap Z_{i-1} \text{ (in the hexagonal case)}$$

and:

$$Z_i \ominus B = (X \ominus (i+1)B) \cup W_i, \text{ with } W_i \cap X \ominus (i+1)B = \emptyset$$

W_i is made of the points of $Z_i \ominus B$ which do not belong to the erosion of size (i+1) of X. V_{i-1} represents the transformation used to ensure the connections of the first kind. We saw that, on the square grid, this transformation may be slightly different. However, in the following proof, the general set V_{i-1} will be used without any reference to its inner transformation, which induces no loss of generality.

Let us show that, for every $j \leq i$, $W_j \subset Z_i$.

Let x be a point of W_j and suppose that this point appeared at step j. This means that x is the center point of the hexagon in the configuration of the set Z_{j-1} drawn at Figure 11a. Otherwise, the point y would be the origin of a configuration belonging to the family **T** and would be suppressed at step j. Then $Z_j \ominus B$ would not contain x.

But as it is the center point of the hexagon, x will never be suppressed by the next thinnings, because, in this case, x should belong to the configuration shown in Figure 11b. However, the point y necessarily belongs to the residues of Z_j because, otherwise, the point labeled 2 would be included in a hexagon and would not be a residue, which is contradictory. So, we have:

$$\bigcup_{j=0}^{i} W_j \subset Z_i$$

Let us calculate now the residue $R(Z_i)$. We have:

$$(Z_i)_B = (Z_i \ominus B) \oplus B = (X \ominus iB)_B \cup (W_i \oplus B)$$

and:

$$Z_i = (Z_{i-1} \ominus B) \cup V_{i-1} = (X \ominus iB) \cup W_{i-1} \cup V_{i-1}$$
$$R(Z_i) = Z_i / (Z_i)_B$$
$$R(Z_i) = [(X \ominus iB) \cup W_{i-1} \cup V_{i-1}] \cap (X \ominus iB)^c_B \cap (W_i \oplus B)^c$$

which can be written:

$$R(Z_i) = [R(X \ominus iB) \cup W_{i-1} \cup V_{i-1}] \cap (W_i \oplus B)^c$$

Given:

$$V_{i-1} = R(Z_{i-1}) \cup Y_{i-1}, \text{ with } Y_{i-1} = [(R(Z_{i-1}) \ominus B)/R(Z_{i-1})] \cap Z_{i-1}$$

Then:

$$R(Z_i) = [R(X \ominus iB) \cup R(Z_{i-1}) \cup W_{i-1} \cup Y_{i-1}] \cap (W_i \oplus B)^c$$

If we develop $R(Z_{i-1})$, we get:

$$R(Z_i) = [R(X \ominus iB) \cap (W_i \oplus B)^c] \\ \cup \\ [R(X \ominus (i-1)B) \cap (W_{i-1} \oplus B)^c \cap (W_i \oplus B)^c] \\ \cup \\ [R(Z_{i-2}) \cap (W_{i-1} \oplus B)^c \cap (W_i \oplus B)^c] \cup \ldots$$

which leads to the following inclusion:

$$R(Z_i) \supset \bigcup_{j=0}^{i} \left[R(X \ominus jB) \cap \left[\bigcup_{k=j+1}^{i} (W_k \oplus B) \right]^c \right]$$

Then:

$$S_c \supset \lim_{i \to \infty} Z_i \supset R(Z_\infty) \supset \bigcup_{j=0}^{\infty} \left[R(X \ominus jB) \cap \left[\bigcup_{k=j+1}^{\infty} (W_k \oplus B) \right]^c \right]$$

But:

$$S_c \supset \lim_{i \to \infty} Z_i \supset \bigcup_{k=0}^{\infty} W_k$$
$$S_c \oplus B \supset \bigcup_{k=0}^{\infty} (W_k \oplus B) \supset \bigcup_{k=j+1}^{\infty} (W_k \oplus B)$$

According to the previous formula, S_c contains every point of the residue $R(X \ominus jB)$ except those which are partly or completely covered by $\left[\bigcup_{k=j+1}^{\infty} (W_k \oplus B) \right]$. But, as the latter set is contained into $S_c \oplus B$, $R(X \ominus jB)$ is also included into $S_c \oplus B$. Finally:

$$S_c(X) \oplus B \supset \bigcup_{j=0}^{\infty} R(X \ominus jB) = S(X) \qquad \text{Q.E.D.}$$

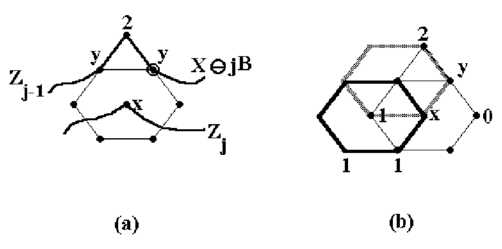


Figure 11: Configurations under study for the set W_j

2.5 Smooth skeletons

As mentioned, the structuring elements family **S** producing a connected skeleton is defined on the neighborhood of size 2 of the origin point. This is necessary if we want to preserve the homotopy by union thinnings. Nevertheless, it is possible to extract from this family **S** a reduced set **S '** of structuring elements preserving the homotopy and defined on the size-1 neighborhood of the origin point. This reduced family is given on the hexagonal grid at figure 12a.

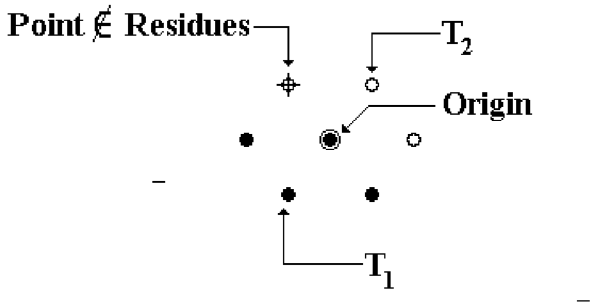


Figure 12: Family of structuring elements producing a smooth skeleton

This family is obtained by splitting the family **S** into two sub-families (Figure 12b) and by keeping only the first configuration which can be reduced to the previous structuring element.

Unfortunately, the union thinnings produced by this reduced family **S '** are far from the previous connected skeleton. However, this transformation is interesting because, if the initial set is regular (this means that X = $(X)_B$), no dendrite appear (Figure 13). Owing to their behavior, these skeletons are called smooth skeletons.

Provided that the initial set is an opened set, these transforms produce "clean" skeletons without dendrites. Consequently, the extremities of these skeletons are significant and their use as shape descriptors can be very helpful.

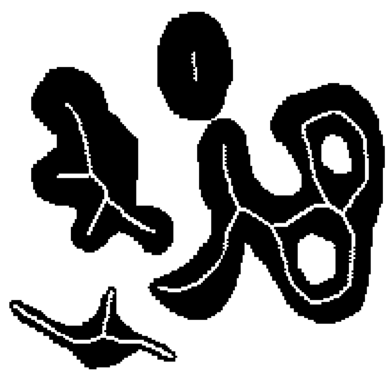


Figure 13: Example of a smooth skeleton

3 Geodesic skeletons

The previous analysis shows that the links between the maximal ball skeleton and the connected skeleton are not so obvious. In particular, it is not simple to build a connected skeleton preserving the good properties of the maximal ball skeleton because the minimum set **S** of the structuring elements which preserve the homotopy whatever the initial set X does not produce a connected skeleton containing the maximal ball skeleton.

However there exists a solution for generating a connected skeleton containing the non connected skeleton. This approach uses the notion of geodesy and of geodesic transformations. In particular, a geodesic skeleton can be defined using geodesic maximal balls then, a geodesic connected skeleton can be designed.

This notion of a geodesic skeleton is very helpful in mathematical morphology. It is the basic brick for building very powerful tools for image segmentation: geodesic skeletons by zones of influence and watersheds [2].

3.1 Geodesic distance, geodesic balls

Let $X \subset Z^2$ be a set, x and y two points of X. We define the geodesic distance $d_X(x,y)$ between x and y as the length of the shortest path (if any) included in X which links x and y . The set X is often called the geodesic space.

A geodesic ball $B_X(x,r)$ of center x and radius r is made of all the points y of X at a geodesic distance $d_X(x,y)$ less than or equal to r:

$$B_X(x, r) = \{y \in X : d_X(x, y) \leq r\}$$

3.2 Geodesic maximal balls

In the Euclidean space, when a ball B_1 with radius r_1 is included in a ball B_2 of radius r_2, we obviously have $r_1 \leq r_2$. In a geodesic space, this relationship is not true as illustrated in Figure 14. The geodesic ball centered in y of radius r contains the geodesic ball centered in z of radius r' > r.

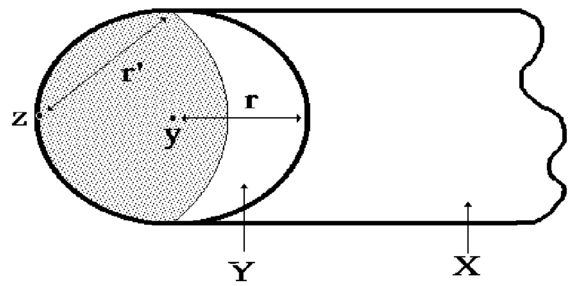


Figure 14: Geodesic maximal balls

Let X be a geodesic space and Y a set included in X. A geodesic ball $B_X(x,r)$ of radius r centered at point x and included in Y is said to be a geodesic maximal ball of Y if and only if there exists no other geodesic ball $B_X(y,r')$ included in Y, <u>with r' > r</u>, such that:

$$B_X(y, r') \supset B_X(x, r).$$

This definition means that, in a geodesic space, a maximal ball may be included in another geodesic ball provided that the radius of the latter ball is less than the radius of the former.

Finally, the geodesic skeleton of Y by geodesic maximal balls can be defined as the set of all the centers of the geodesic maximal balls of Y.

3.3 Geodesic transformations

Some basic geodesic transformations can be defined in a geodesic space X. These transformations use geodesic balls as structuring elements. The geodesic erosion and dilation of a set $Y \subset X$ by a geodesic ball B_X of radius r are given by [6]:

$$E_X^r(Y) = \{y \in X : B_X(y, r) \subset Y\}$$

and:

$$D_X^r(Y) = \{y \in X : B_X(y, r) \cap Y \neq \emptyset\}$$

The geodesic erosion and dilation with an elementary ball B_X of radius 1 can be obtained with the Euclidean operators. We have:

$$E_X(Y) = [(Y \cup X^c) \ominus B] \cap X$$
$$D_X(Y) = (Y \oplus B) \cap X$$

B is the elementary ball in the Euclidean space.

Geodesic hit-or-miss transformations can also be defined. But, in this case, the structuring element $T=(T_1, T_2)$ is defined in terms of geodesic distance and balls. For example, the following structuring element $T=(T_1, T_2)$ is allowed:

$$T_1 = \{B_X(y, r_1), d_X(y, x) \leq l_1\}$$
$$T_2 = \{B_X(z, r_2), d_X(z, x) \leq l_2\}$$

The point x is its origin.

The geodesic hit-or-miss transformation is given by:

$$HMT_X(Y; T) = E_X(Y; T_1) \cap E_X(X/Y; T_2)$$

with:

$$E_X(Y; T_1) = \{z \in X : \exists\ y, d_X(y, z) \leq l_1 \text{and } B_X(y, r_1) \subset Y\}$$

The geodesic thinning is given by:

$$Thin_X(Y; T) = Y/HMT_X(Y; T)$$

3.4 Geodesic skeleton by thinnings

The geodesic maximal ball skeleton of a set Y in a geodesic space X can be built by adding the residues of the successive geodesic erosions of Y. We have (Figure 15a):

$$S_X(Y) = \bigcup_i R(E_X^i(Y))$$

with:

$$R(E_X^i(Y)) = Y/[D_X(E_X^{i+1}(Y))]$$

This skeleton may also be obtained by successive geodesic thinnings with a structuring element $T=(T_1, T_2)$ given by (in the digital case):

$$T_1 = \{B_X(y, 1), d_X(x, y) \leq 1\}$$
$$T_2 = \{y : d_X(x, y) \leq 1\} \qquad \text{(x is the origin of T)}$$

We have:

$$S_X(Y) = \lim_{n \to \infty} \{Thin_X(Y; T)\}^n = \lim_{n \to \infty} (Thin_X(Thin_X(...(Thin_X(Y; T); T); T); T))$$

with:

$$Thin_X(Y; T) = Y/[D_X(E_X(Y)) \cap D_X(X/Y)]$$

D_X and E_X are respectively the elementary geodesic dilation and erosion.

By using the definition of these elementary transformations in terms of Euclidean operators (see above), we find:

$$Thin_X(Y; T) = Y \cap \left[\bigcup_{a,b \in B} [(Z \ominus B_a) \cap (Z^c \ominus L_b) \cap (X \ominus L_a)] \right]^c$$

with $Z = Y \cup X^c$.

This thinning almost corresponds to the thinning of Z by the family T of structuring elements. The only difference for the geodesic non connected skeleton is the term $(X \ominus L_a)$.

3.5 Connected geodesic skeleton

From this non connected geodesic skeleton, the design of a connected version is possible by selecting among the family T of structuring elements those which preserve homotopy. In particular, in the case of the hexagonal grid or of the square 8-connected grid, we saw that T2 is reduced to one point so that the preceding formula becomes:

$$\mathrm{Thin}_X(Y;T) = Y \cap \left[\bigcup_{a \in B} [((Y \cup X^c) \ominus B_a) \cap ((Y \cup X^c)^c \ominus L_a)] \right]^c$$

because the only L_b used is L_a.

In this case, the selected thinning exactly corresponds to the thinning of $(Y \cup X^c)$ by the family **T''** of structuring elements which preserve the connections of the second kind in the Euclidean connected skeleton.

$\mathrm{Thin}_X(Y;T) = (Y \cup X^c) \bigcirc \mathbf{T''}) \cap X$ with $\mathbf{T''} = \{(B_a, L_a), \forall a \in B\}$

We have now to take into account the residues of the geodesic opening. The algorithm for the connected skeleton $Sc_X(Y)$ finally is (Figure 15b):

1. $Y_0 = Y$
2. Calculation of $Z_i = (Y_i \cup X^c) \bigstar \mathbf{T''}$
3. Residues R_i of Y_i by geodesic opening:
$$R_i = Y_i / [(((Y_i \cup X^c) \ominus B) \cap X) \oplus B]$$
4. Suppression of the points of Z_i adjacent to the residues:
$$Z_i = Z_i / (R_i \oplus B) \qquad \text{(in the case of the hexagonal grid)}$$
5. Calculation of $Y_{i+1} = Y_i / Z_i$
6. If Y_{i+1} and Y_i are different, return to (2), otherwise $Sc_X(Y) = Y_{i+1}$

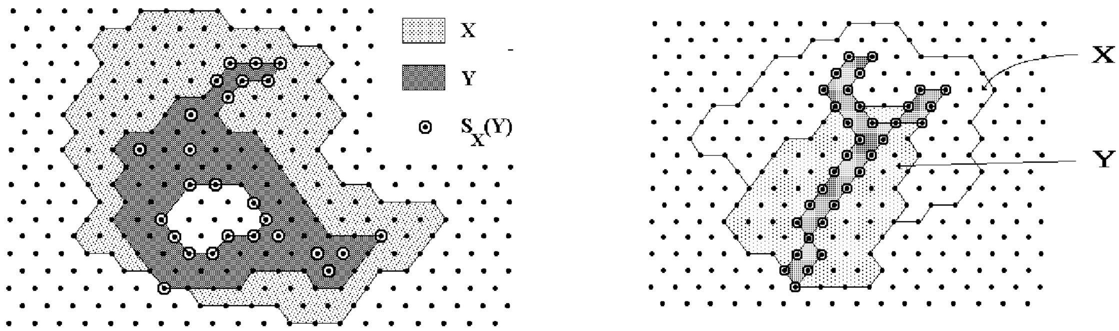


Figure 15: Non connected (a) and connected (b) geodesic skeletons

3.6 Connecting the maximal ball Euclidean skeleton by means of geodesic thinnings

This geodesic connected skeleton can be used for connecting the maximal balls skeleton. Let X be a set in Z^2. We can first calculate its maximal ball skeleton S(X). Then, this skeleton can be subtracted from X. The resulting set Y is:

$$Y = X/S(X)$$

Then, the geodesic connected skeleton of Y is performed. The geodesic space used is $Z^2/S(X)$ and is denoted Z. Finally, the maximal ball skeleton S(X) is added to the result to produce a connected skeleton $S_c(X)$ containing the maximal ball skeleton:

$S_c(X) = Sc_Z(X/S(X)) \cup S(X)$ with $Z = Z^2 \cap [S(X)]^c$

However, this procedure is rarely used in practice because it first requires to compute the maximal ball skeleton which makes it too long. Moreover, the errors introduced in the Euclidean connected skeleton are not very important (in many cases, actually, the connected skeleton contains the maximal balls skeleton).

Conclusion

The definition of good skeletonization algorithms is not a simple task. We have seen that the making of a connected skeleton preserving the good properties of the maximal ball skeleton is not obvious. Using union thinnings with a family of structuring elements which preserve homotopy does produce a unique skeleton but this skeleton unfortunately does not contain the centers of the maximal balls. However, we have proved that these centers are always adjacent to this connected skeleton.. This is why the computation, by means of geodesic thinnings, of a connected skeleton passing through the maximal balls skeleton is not used in practice.

One may wonder whether if the "skeletons" using sequential thinnings would not be sufficient. In fact, it is not the case because the resulting transform is often biased. This bias also appears in the skeleton by zones of influence and in the watershed transformation, both transformations being built with connected skeletons are very useful in binary and gray tone image segmentations.

The skeleton transformation is not very robust, even when the initial set is regular. Therefore the skeleton of a set is not a good shape descriptor. But the smooth skeleton defined by reducing the structuring elements family of the connected skeleton, often provides a better result. This is one of the valuable consequences of this analysis of the connected skeletons in digital spaces.

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