ISMM05 Special Issue

Numerical Residues

Serge BEUCHER
Centre de Morphologie Mathématique
Ecole des Mines de Paris
35, rue Saint Honoré
77305 FONTAINEBLEAU (France)
(*Serge.Beucher@ensmp.fr*)

Abstract

Binary morphological transformations based on the residues (ultimate erosion, skeleton by openings, etc.) are extended to functions by means of the transformation definition and of its associated function based on the analysis of the residue evolution in every point of the image. This definition allows to build the transformed image and its associated function, indicating the value of the residue index for which this evolution is the most important. These definitions have the advantage of supplying effective tools for shape analysis and of allowing the definition of new residual transforms together with their associated functions. Two of these numerical residues will be introduced, called respectively ultimate opening and quasi-distance and, through some applications, the interest and efficiency of these operators will be illustrated. Finally, this residual approach will be extended to more complex operators.

Keywords

Image analysis, mathematical morphology, residues, ultimate opening, quasi-distance, image segmentation, hierarchy.

1. Introduction

In binary morphology some operators are based on the detection of residues of parametric transformations. Among these operators, the ultimate erosion or the skeleton by maximal balls can be quoted. They can more or less easily be extended to greytone images. These extensions are however of little use because it is difficult to exploit them. This paper explains the reasons of this difficulty and proposes a way to obtain interesting information from these transformations. It also introduces new residual transformations and illustrates their use in applications.

2. Binary residues: reminder of their definition

Only operators corresponding to the residues of two primitive transforms will be addressed here. A residual operator θ on a set X is defined by means of two families of transformations (the primitives) depending on a parameter i ($i \in I$), ψ_i and ζ_i, with $\psi_i \geq \zeta_i$. The residue of size i is the set:

$$r_i = \psi_i \setminus \zeta_i$$

the transformation θ is then defined as:

$$\theta = \bigcup_{i \in I} r_i$$

Usually, ψ_i is an erosion ε_i. According to the choice of ζ_i, we get the different following operators:

- The ultimate erosion [3]; the operator ζ_i is then the elementary opening by reconstruction of the erosion ε_i:

$$\zeta_i = \gamma_{rec}(\varepsilon_i)$$

- The skeleton by maximal balls; in that case the operator ζ_i is the elementary opening of the erosion of size i:

$$\zeta_i = \gamma(\varepsilon_i)$$

Generally a function q, called associated function is linked to these transformations. The support of q is the transformed $\theta(X)$ itself. This function takes in every point x, the value of index i of residue r_i containing point x (or more exactly the value i+1, so that this function is different from zero for r_0). Indeed, in the binary case, if the primitives are correctly chosen, to every point x corresponds a unique residue. Then, we get:

$$q(x) = i + 1 : x \in r_i$$

For the ultimate erosion, this function corresponds to the size of the ultimate components. For the skeleton, it is called *quench function* and corresponds to the size of the maximal ball centered in x.

3. Extension to greytone images

Although it is common to read or to hear that these operators can be extended without any problem to the numerical case (greytone images), this extension is in fact not as straightforward as it appears to be, for the transformation θ itself but also and especially for the associated function q.

3.1. Definition of the operator θ in the numerical case

A "simple" definition of θ can be written:

$$\theta = \sup_{i \in I}(\psi_i - \zeta_i)$$

by using the numerical equivalents of the set union and difference operators.

However by doing so, a first problem appears. The subtraction of functions is not really equivalent to the set difference. In the binary case, we had, for a sensible choice of the primitive:

$$\forall i, j\ i \neq j : r_i \cap r_j = \varnothing$$

In the numerical case, it is not true any more. The residues r_i and r_j may have a common support, which entails that $\inf(r_i, r_j) \neq 0$ (Figure 1).

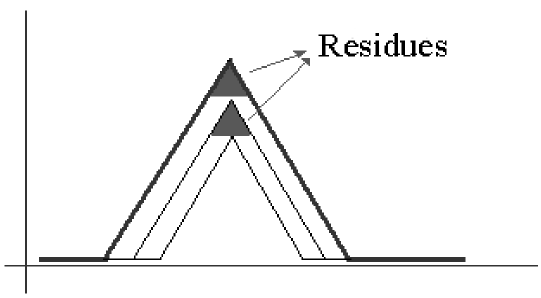


Figure 1: Superimposed residues appear for various sizes of erosion

It follows that, in the numerical case, the definition of function q associated to transformation θ is not as evident as in the binary case where every r_i has a different support.

3.2. Definition of a simplified q function

Let us define a simplified q function by observing the construction of the transformed function θ and the evolution of this construction in every point x of the domain of definition of the initial function f. To do so, let us come back to the design of the transformations in the binary case by replacing set X by its indicator function k_X and by observing how the indicator function $k_{r_i}(x)$ of the residues at point x evolves for each transformation step.

In the case of an indicator function, this evolution is obvious: all $k_{r_j}(x)$ are equal to zero except the indicator function of the residue r_i containing x. It can be written:

$$q(x) = i+1 : k_{r_i}(x) \neq 0$$

if we replace the indicator of X by a two-level function (with b<a),

$$f(x) = a \text{ if } x \in X$$

$$f(x) = b \text{ if not}$$

the phenomenon does not change (Figure 2).

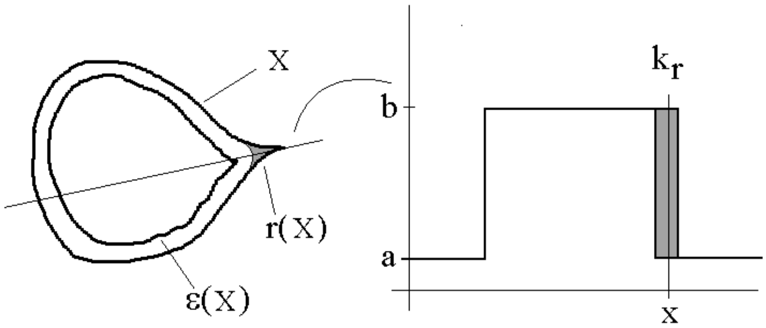


Figure 2: Residue and its indicator function

Let us take now the case of a general function f. In that case, there are several values of index i for which the difference $r_i(x) = \psi_i(x) - \zeta_i(x)$ at point x is different from zero.
So, in order to keep the most significant residue, we define an associated function q with a value at point x equal to index i for which $r_i(x)$ is positive and maximal.

$$q = \max(r_i) + 1 = \max(\psi_i - \zeta_i) + 1$$

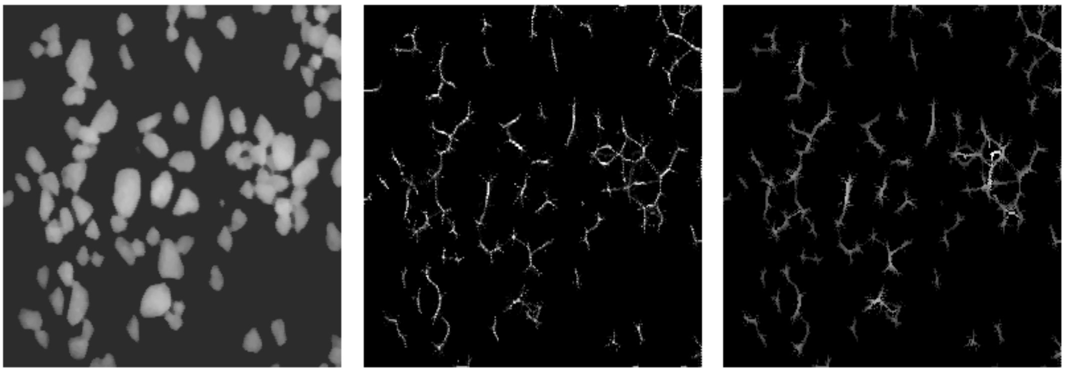
$$\{q(x) = i+1 : r_i(x) > 0 \text{ and maximal}\}$$

If this maximum appears for several values of i, only the highest value will be retained:

$$q(x) = \{\max(i) + 1 : r_i(x) > 0 \text{ and maximal}\}$$

The ultimate erosion obtained by applying these definitions is illustrated below (Figure 3).

Notice that, when the original image is more or less a two-level one, the result is not very different from the one obtained by using the binary versions of these operators on a thresholded image. The advantage of the approach is to avoid this thresholding step (which in this particular case could be problematic).



Initial image *Ultimate erosion* *Associated function*

Figure 3: Greytone ultimate erosion for a greytone image

Obviously, it is not possible any more to entirely reconstruct the initial image from its skeleton and the associated function as it was in the binary case. One can however define a partial reconstruction ρ(f) of the initial function f as follows:

$$\rho(f) = \sup_{x \in E}\left(\theta(x) \oplus B_{q(x)}\right)$$

At every point x a cylinder is implanted, its base being a disc with a radius equal to the value of the associated function in this point and its height being given by the value of the residue at the same point (Figure 4).

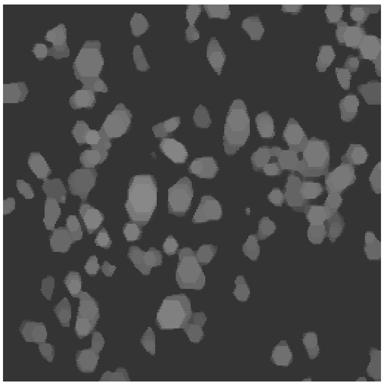


Figure 4: Image reconstructed from the numerical skeleton by openings

4. New operators

All previous residues are residues of differences of erosions and openings. However many other operators can be defined in binary as well as in numerical cases from different primitive transformations ψ_i and ζ_i. Indeed it is enough that they depend on a parameter i and that they verify the relation $\psi_i \geq \zeta_i$ to be "eligible". However, many of these transformations seem to be of low interest because the results obtained are either too simple, or available by simpler means. Nevertheless, some operators are really interesting. Some, indeed, provide self-evident residues but are far from being uninteresting when the associated function is considered. Others, while presenting low interest in binary, become very useful for greytone images. To illustrate this, let us introduce two new residual operators named respectively *ultimate opening* and *quasi-distance*.

4.1. Ultimate opening

Let us consider the residual operator ν where the primitives ψ_i and ζ_i are respectively an opening by balls of size i and an opening by balls of size i+1:

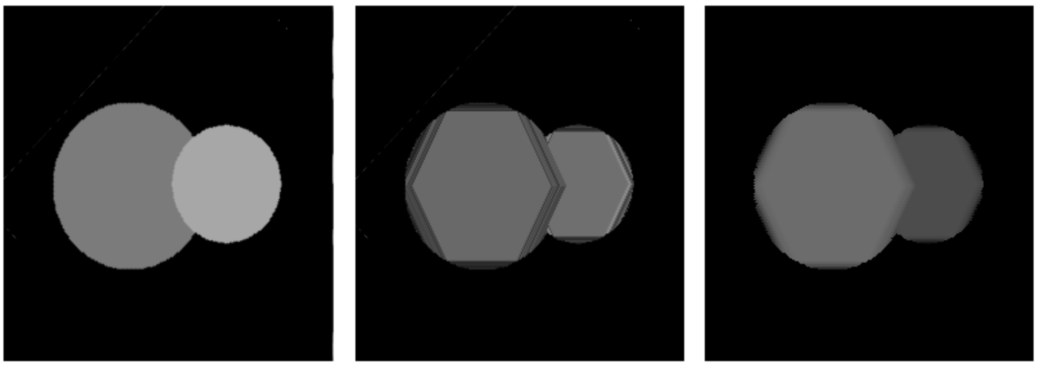
$$\psi_i = \gamma_i$$
$$\zeta_i = \gamma_{i+1}$$
$$\nu = \sup_{i \in I} (\gamma_i - \gamma_{i+1})$$

The operator ν does not present any interest in the binary domain. Indeed, in that case, it is easy to show that it is equal to the identity. In the numerical domain, it replaces the initial image by a union of the most significant cylinders included in the sub-graph of the initial function.

A significant cylinder is the biggest and highest cylinder included in the sub-graph of the initial function. It is the biggest cylinder covering every point of the image.

This operator does not provide the same result as the reconstruction described previously. It emphasizes the size of the significant cylinders as illustrated in Figure 5.

The associated function s, even in the binary case, presents a higher interest. In every point x, s(x) is equal to the size of the biggest disk covering this point x (binary case) or to the radius of the biggest significant cylinder of the partial reconstruction covering (numerical case). Function s is called *granulometric function.*



Initial image *Ultimate opening* *Granulometric function*

Figure 5: Ultimate opening and granulometric function

4.2. Quasi-distance

In the previous definition, openings can be replaced by erosions. A new residual operator τ is then defined; its interest also lies in its associated function.

$$\psi_i = \varepsilon_i$$
$$\zeta_i = \varepsilon_{i+1}$$
$$\tau = \sup_{i \in I} (\varepsilon_i - \varepsilon_{i+1})$$

In the binary case, this operator is of no interest because it is equal to the identity and its associated function d is nothing else than the distance function.

In the numerical case, the physical interpretation of the residue itself is not very explicit. The associated function d is more interesting: it is very close to a distance function calculated on the significant flat or almost flat zones of the initial function. By significant, one means a zone corresponding to an important variation of the erosion.

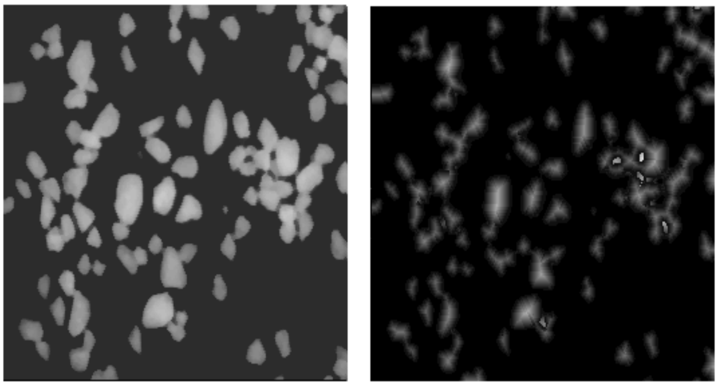


Figure 6: Quasi-distance (right) of the initial image (left)

Figure 6 shows this transformation applied to an almost two-level image. Even on this relatively simple image, in certain places the appearance of rather high values of this quasi-distance can be noticed. These values come from the erosion of relatively flat zones which appear when zones above have been eroded and have disappeared (Figure 7). They correspond to "hung up" distance functions. When the initial function is arranged in terraces (flat zones which are not extrema), its quasi-distance is not symmetric on the flat zones which do not correspond to maxima.

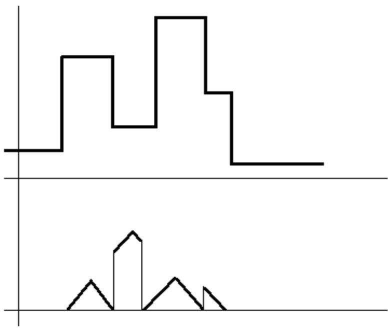


Figure 7: Multi-level function and its quasi distance

Different strategies can be used to correct this phenomenon. One promising technique consists in looking for the zones where the quasi-distance is not 1-lipschitzian and to correct these zones by an iterative approach, as shown next.

4.3. Corrected quasi-distance

A classical distance function d is 1-lipschitzian. It means that, given two points x and y, the following relation holds:

$$|d(x) - d(y)| \leq \|x - y\|$$

In particular, when x and y are two adjacent digital points, their distance is at the most equal to 1. It is obviously not the case for the quasi-distance due to the "hung up" distances and to the non symmetric distances on some plateaus. It is however possible to force this quasi-distance to be 1-lipschitzian by means of an iterative procedure of "descent of hung-up distances". It consists in subtracting from the function d distances larger than 1 between a point and its neighbours (ε denotes the unit erosion).

- For any point x where $[d - \varepsilon(d)](x) > 1$, do $d(x) = \varepsilon(d)(x) + 1$
- The procedure is iterated until idempotence.

Figure 8: Quasi-distance before and after correction.

5. Applications

In the same way as the residues were used in set segmentation, their numerical versions as well as the new residues described above constitute remarkable tools of granulometric description and of markers generation for segmentation. To illustrate the potentialities of these transformations, let us present three applications in greytone segmentation.

5.1. Critical disks

In this first application, the ultimate opening and its associated function are used to extract the critical disks of a binary set X.

The notion of critical disk (or ball) was introduced in [8] and [5]. Our purpose, then, was to enhance the segmentation of binary particles when they were too intricated to be separated by ultimate erosions. The skeleton by maximal disks of a set X corresponds to the centers of the maximal disks included in X. A maximal disk is a disk which is not included in any other disk contained in X. It is known also that the set X can be totally reconstructed by means of its maximal disks. However, this set of maximal disks is redundant for the reconstruction. In many cases, a subset of the maximal disks is sufficient to reconstruct the initial set. The maximal disks belonging to this subset are called c*ritical disks.* A maximal disk is critical when it cannot be covered with any combination of the other maximal disks.

In R^n, one can show easily that, if we consider euclidean disks (balls), the set of critical disks of X is unique. However, if the disks are polyedra ((hexagons for example in R^2 or in digital spaces), this uniqueness is not verified. In that case, several covering solutions may appear. It is due to the fact that the union of maximal polyedral disks of same size may generate new maximal disks of same size (Figure 9).

Indeed, the skeleton of a set is generally a bad shape descriptor, due to the occurrence of these non critical redundant disks. Many solutions have been proposed to define minimal skeletons [7, 10]. The multiple possibilities of covering induced by multiple choices of critical disks explain the complexity of the problem.

However, the use of the function associated to the ultimate opening together with a restricted definition of a critical disk allows to filter some maximal disks and to produce a skeleton which is not minimal but nevertheless leads to a better separation of deeply intricated particles.

In the digital case, a restricted definition of a critical disk is given by:

$$B_i \text{ critical : it does not exist } J = \{j_1, \ldots, j_N : j_k \neq i\} \text{ such that } B_i \subset \bigcup_{j \in J} B_j$$

A digital maximal disk B_i of size i is critical if there is no combination of maximal disks B_j, the size j of B_j being different of the size i of B_i, which covers B_i.

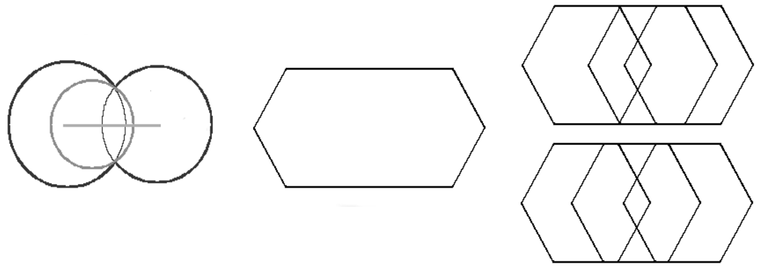
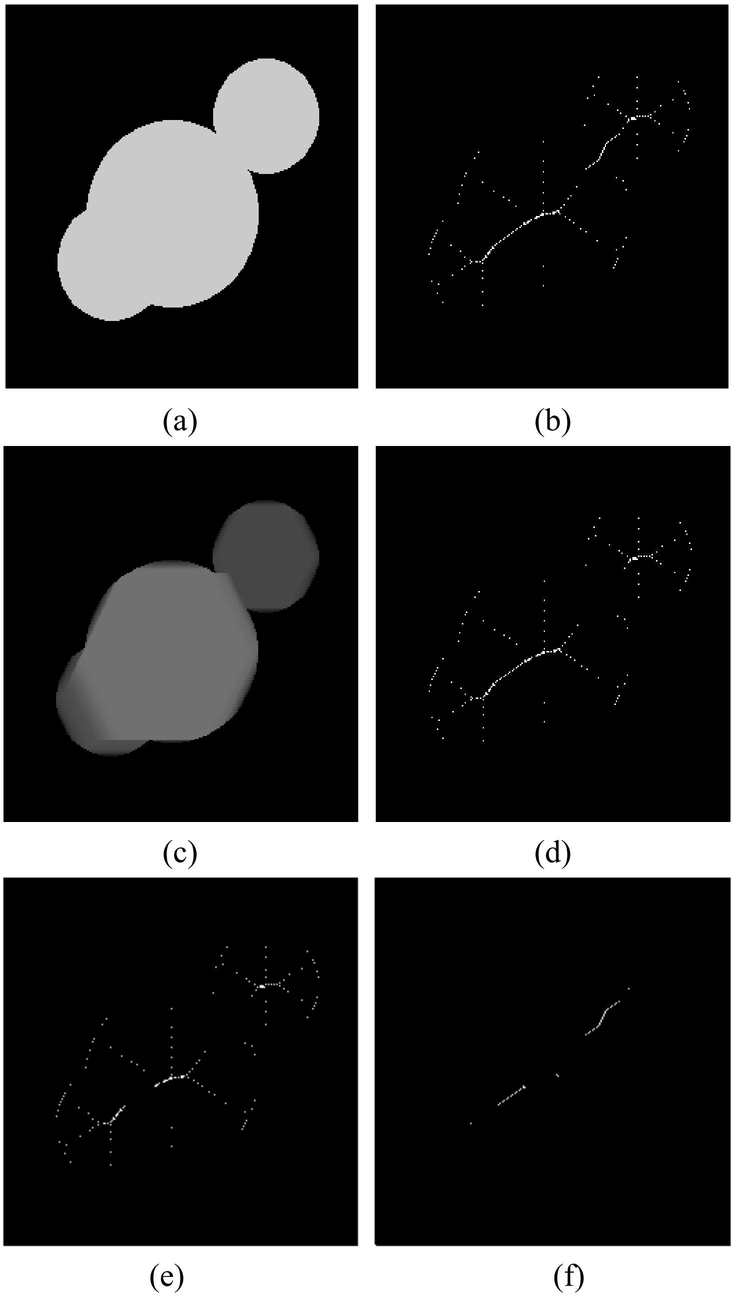


Figure 9: On the left, the inside disk is not critical. On the right, two coverings of the initial set by different "critical" hexagons

The function associated to the ultimate opening is, in the binary case, a first filter which eliminates some non critical disks. Indeed, given that only the biggest disks appear in this associated function, every maximal disk covered with disks of bigger size is suppressed. Furthermore, only the parts uncovered with disks of bigger size are visible in those disks that were preserved. So, a second filter can be applied on this associated function. It consists in detecting lower size disks which are able to cover what was uncompletely covered with higher size disks.

The procedure is as follows (Figure 10):

- Starting from the skeleton quench function q and from the granulometric function s associated with the ultimate opening, for every level i, it is verified that the centers of the maximal disks of size i are covered. By considering $Z_i = \{x : q(x) = i\}$, the centers of the maximal disks of size i-1 and $Y_i = \{y : s(y) = i\}$, the set of the points of these uncovered disks, the intersection $Z_i \cap \delta_i(Y_i)$ (δ_i is a dilation of size i) provides the centers of the potentially critical disks of size i-1.
- The second step consists in verifying if the potentially critical disks selected by the first step cover or not some potentially critical disks of bigger size. For this, we calculate the infimum between the granulometric function and every cylinder of size i and of height i+1 implanted in every centre of potentially critical disk of radius i obtained during the first stage. The grey levels of the resulting function mark uncovered disks by sets of bigger or/and smaller disks, in other words the critical disks. The centers of these disks can be then built by the same procedure as the one used in the first stage.



<u>Figure 10</u>: The various stages of the selection of the critical disks
(a) Initial set, (b) centers of maximal disks, (c) granulometric function
(d) centers of maximal disks not covered with bigger disks
(e) centers of critical disks and (f) of non critical disks

Starting from the critical disks, a new granulometric function can be defined by performing the union of the cylinders associated to these critical disks. This function can itself be used to generate markers. For example, the significant critical disks can be extracted i.e, those that are sufficiently uncovered. From these first markers, one selects the centers of the corresponding critical disks which can be used as second markers (Figure 11).

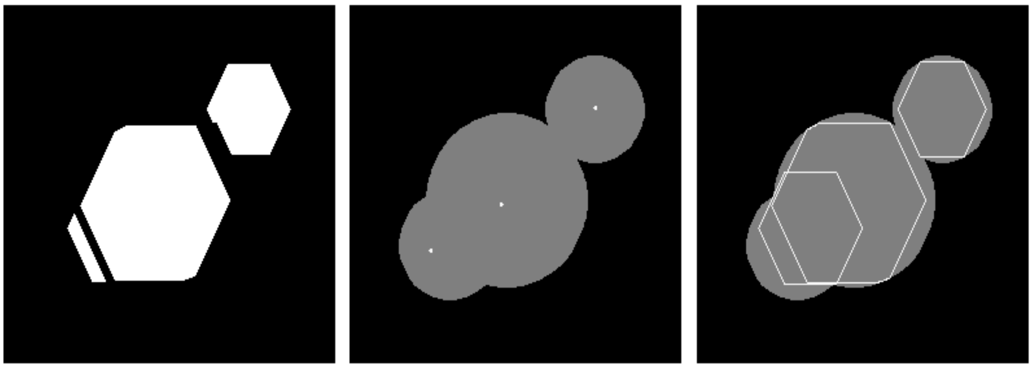
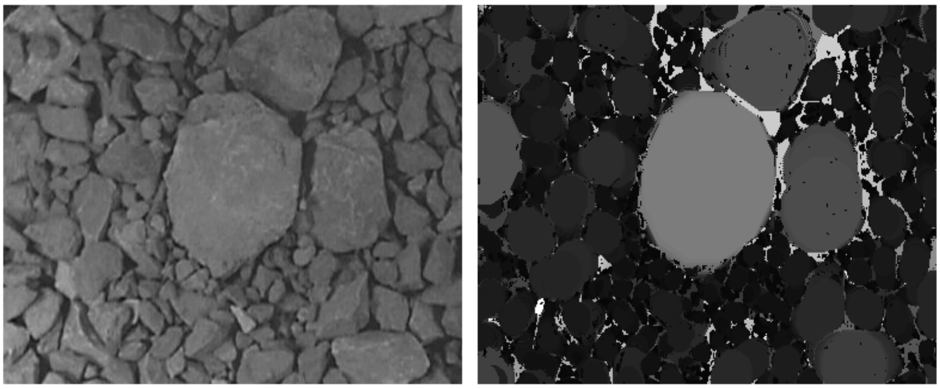


Figure 11: : Selection of the most salient critical disks (in fact hexagons) of the image

5.2. Size distribution and segmentation of blocks

An application of granulometric functions consists in defining real size distributions of objects in an image without the necessity of extracting them beforehand. Furthermore, this size distribution is always closer to the real size distribution of the analyzed objects than the one obtained by successive openings of the image.



(a) *(b)*

Figure 12: Blocks of rocks (a)(© CGES/ENSMP) and associated granulometric function (b)

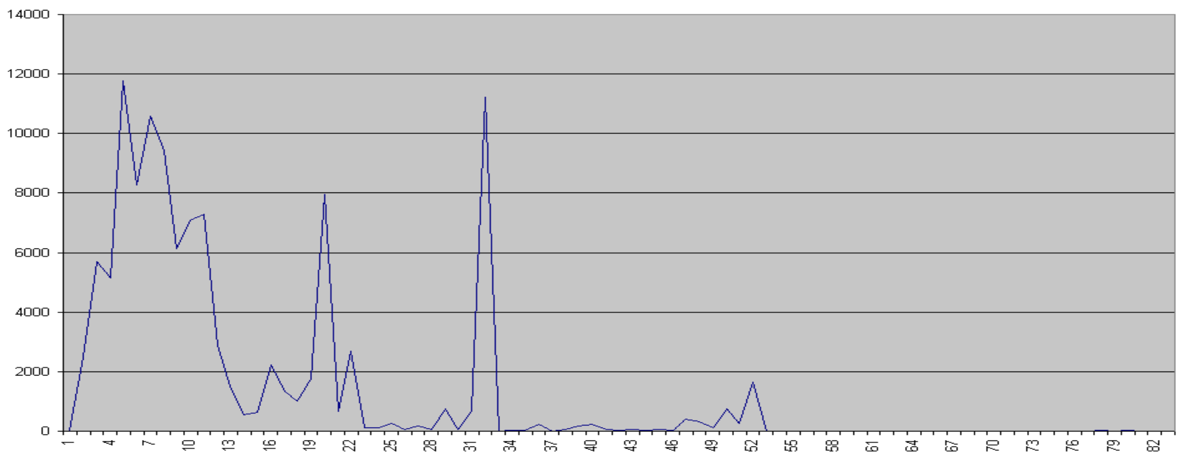


Figure 13: Size distribution of blocks (histogram of the granulometric image)

Figure 12a represents a heap of rocks. A granulometric function can be built, associated to the ultimate opening (Figure 12b, openings are isotropic). As the value of every pixel corresponds to the size of the biggest opening containing this pixel, the histogram of the granulometric function (Figure 13) produces a size distribution curve very close to the real size distribution of blocks (at least in 2D).

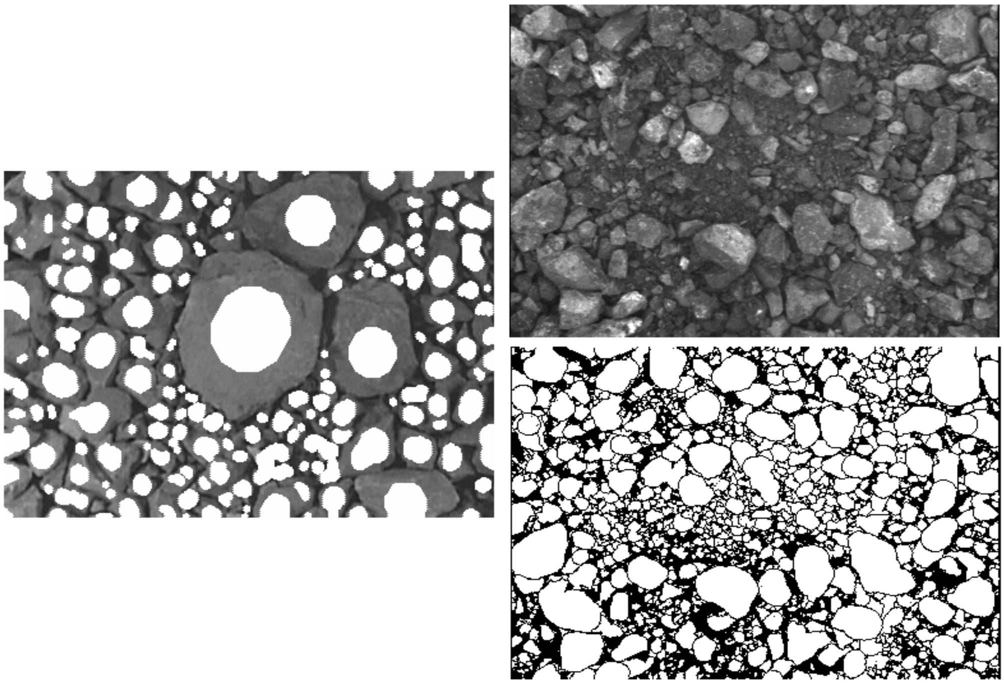


Figure 14: Blocks marking and segmentation

Granulometric functions can also be used to mark blocks. Then, markers can be counted or used to perform segmentations of the image by watersheds. The generation of these markers is made by performing on every threshold of the granulometric function an erosion of a size proportional to the threshold value. Figure 14 illustrates these algorithms On the left, markers of the blocks of rocks have been defined by this size-controlled erosion. On the right, the result of the marker-controlled gradient watershed of the initial image (upper view) is shown (lower view).

5.3. Image segmentation

The third application will use quasi-distances. This application is only a sketch of the possibilities offered by this type of tool.

One saw previously that the quasi-distance allows to build a distance function for the relatively flat and relevant zones of a greytone image. This property is used here to exhibit the markers of these regions. Then these markers can be used to control the watershed transformation of the quasi-distance in order to segment the homogeneous regions of the image. The various steps of the algorithm are the following (Figure 15):

- Computation of the quasi-distance of the initial image f.
- Image inversion and computation of the quasi-distance of f^c.
- Supremum of the two quasi-distances.
- A threshold of this new function at a given level i allows the extraction of homogeneous regions of the image of size larger than i.
- Computation of the watershed transform of the supremum controlled by the previous

markers.

The calculation of the quasi-distance of the inverted image is compulsory to exhibit the dark regions which can correspond to minima of the image. One saw previously that, in that case, quasi-distances are either equal to zero or "hung up". The calculation of this quasi-distance after inversion allows to take into account the real sizes of these structures. Notice also that this segmentation does not use the image gradient.

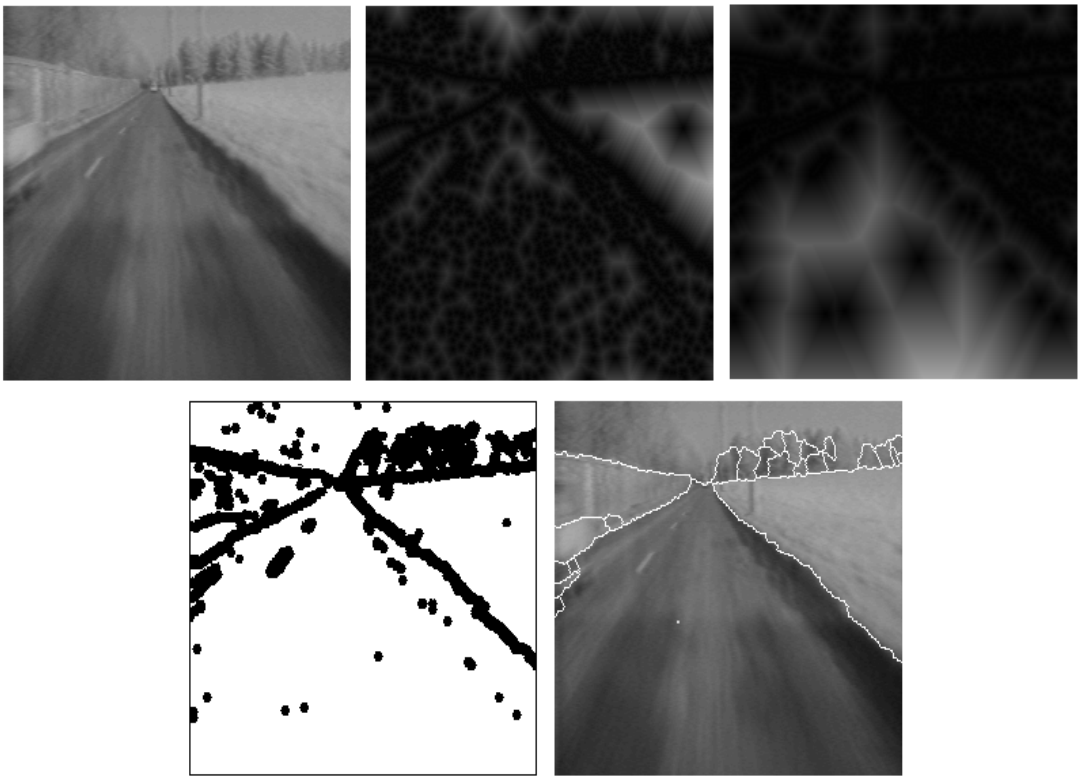


Figure 15: Use of quasi-distances in image segmentation

6. Extending the concept of residues

It is very common, in mathematical morphology, to use parametric transforms. However, the value of these parameters (size of a filter, level of a hierarchy, threshold value, etc.) is sometimes difficult to determine. The best value may vary from image to image and even inside a given image.

The notion of residue, by indicating the parameter value corresponding to the strongest effect of a parametric transformation, can be very efficient in the process of selecting these best values. In fact, this capability has already been illustrated in the block marking and segmentation example: the size of the erosion applied for extracting the markers is locally determined according to the granulometic function.

Residues can be derived from many initial transformations ψ_i and ζ_i. It is not compulsory that these transformations are simple. In order to illustrate the efficiency of this notion of residues, let us apply this concept to define a new morphological hierarchical segmentation tool.

The waterfall transformation is often used to produce various levels of hierarchical segmentation from an initial watershed transform [1, 6, 9]. Starting from a greytone image f, we compute its gradient g, then the valued watershed of g, denoted w_0. By means of a dual geodesic reconstruction of w_0, we get a new image, called hierarchical image. The watershed of this hierarchical image gives a new segmentation where the less significant contours have been removed. The process can be iterated from this new watershed to produce successive levels of hierarchy. This sequence of waterfall transformations, although very efficient in some cases, presents various drawbacks. Firstly, it is impossible to determine which level of hierarchy corresponds to the best segmentation. Secondly, this best segmentation does not occur for the same level of hierarchy everywhere in the image. However, these two problems can be solved by means of a residual approach.

Let us define iteratively the transformation ψ_i from the previous transform ψ_{i-1}. Let m_{i-1} be the minima of ψ_{i-1}. Let us build a new function ξ_{i-1} equal to ψ_{i-1} everywhere except for those points belonging to the minima where $\xi_{i-1} = +\infty$ (in practice, ξ_{i-1} takes the maximal possible grey value of the image). Then, we can define ψ_i as the dual geodesic reconstruction R^* of ψ_{i-1} using ξ_{i-1} as marker:

$$\psi_i = R^*_{\varsigma_{i-1}}(\psi_{i-1})$$

By definition $\psi_0 = w_0$.

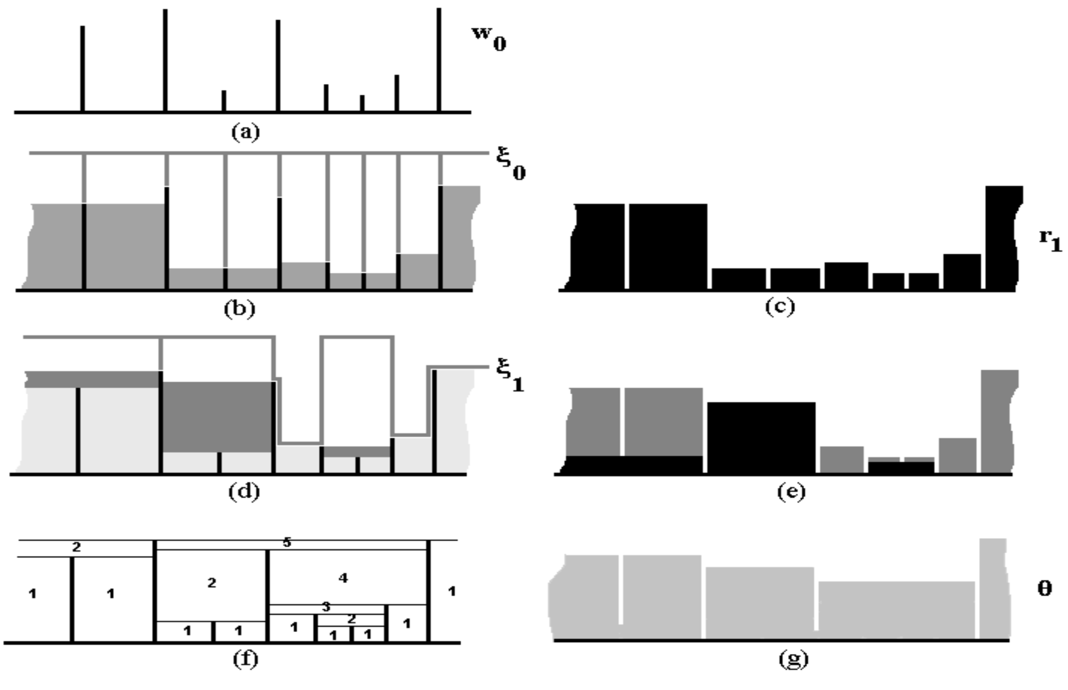


Figure 16: Hierarchical segmentation from residues of pilings
(a) Initial watershed, (b) construction of ψ_1, (c) first residues, (d) construction of ψ_2 (in grey) and residues r_2 (dark grey), (e) supremum of r_1 and r_2, (f) the various piles and their order of appearance, (g) final θ function (supremum of all residues)

Figure 16 illustrates the transformation. This operator fills in the catchment basins associated to the minima m_i with successive pilings. We define now the residue r_i as follows:

$$r_i = (\psi_i - \psi_{i-1})$$

Each residue corresponds in fact to the new piles which have been added to fill in the catchment basins of the initial watershed transform. Then, we define the two functions θ and q from the residues:

$$\theta = \sup_{i \in I} (\psi_i - \psi_{i-1})$$
$$q = \arg\max(r_i) = \arg\max(\psi_i - \psi_{i-1})$$

The function θ, by removing the contours which are covered by thicker piles, preserves regions in the image surrounded by significant contours. A contour is significant when the contrast of all the contours inside the enclosed region is less than half the contrast of the significant contour. It is interesting to note that this approach works even when the significant region does not present internal contours (this is not true with the waterfall algorithm). The contours which remain in θ (hollow contours) correspond to the best hierarchy of segmentation.

The associated function q is an indicator of the homogeneity or complexity of each preserved region. The lower the value of q, the smoother the corresponding region. Figure 17 shows the efficiency of this operator on a real image.

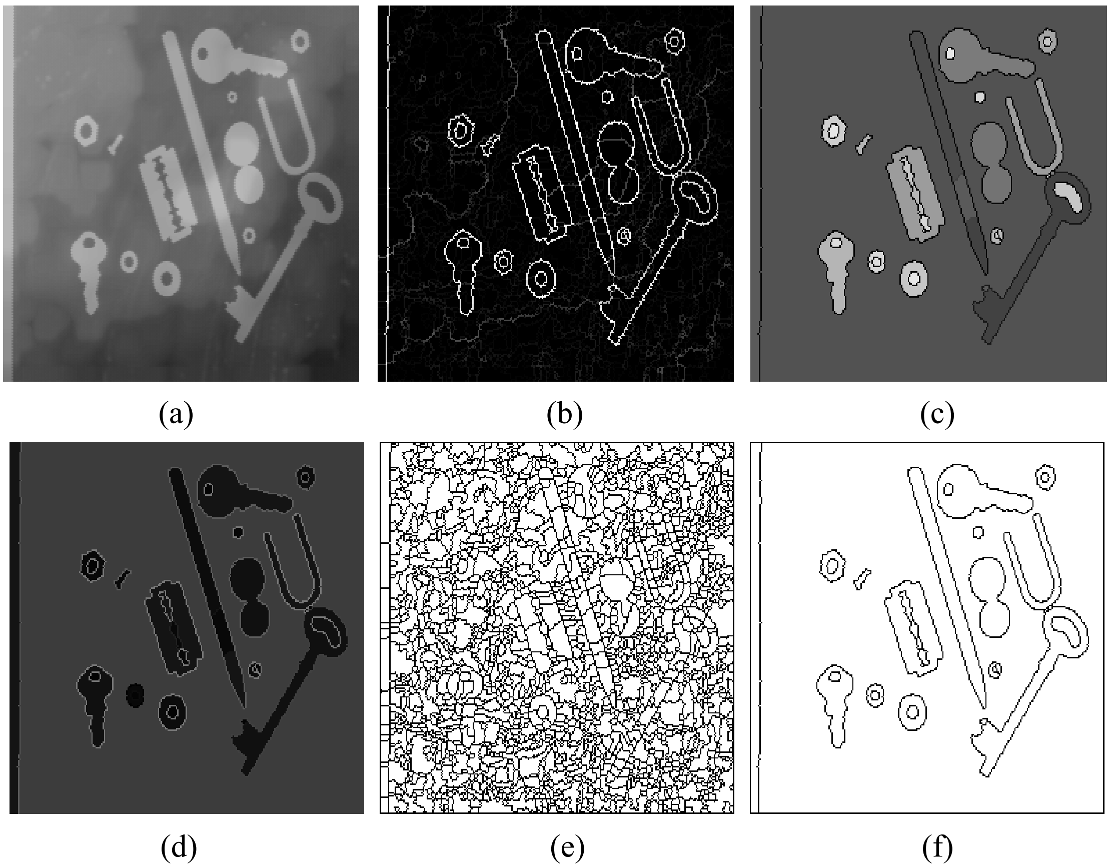


Figure 17: Initial image (a), valued watershed of gradient (b), supremum of pilings θ (c) Associated function q (d), initial contours (e), preserved contours after piling process (f)

7. Conclusions

Not only does the definition of numerical transformations based on residues provide extensions of efficient tools in binary and numerical morphology but, furthermore, allows to introduce new operators whose potentialities are enormous. The importance of the doublet constituted by the transformation and by its associated function has also been emphasized, this last one being sometimes more interesting in numerical morphology than in binary one.

The extension of these notions and especially the definition of a simplified associated function was possible by changing our point of view: rather than focusing on the neighborhood relationships between the image points, we point out the modifications which occur vertically in a point as we "unwind" the transformation, the most significant changes and especially the moment when they occur constituting the core of information provided by these operators.

The applications presented as illustrations of the potentialities of the granulometric functions and of the quasi-distances still deserve additional developments. However, the efficiency of these operators can already be verified and a large number of tracks of future applications can be considered.

The granulometric function is a powerful segmentation and filtering tool. By associating every point of the image to the size of the highest cylinder included in the sub-graph, it allows ipso facto to adapt the size of the filters which are applied in each of its thresholds. It is also possible to eliminate too deeply covered components or, on the contrary, to extract non covered blocks. This capability is interesting in numerous applications where objects appear in heap and where random sets models ("dead leaves" models notably) are used. The topology of every threshold of the granulometric function and in particular the presence of holes is very important. These holes indicate generally the presence of superimposed structures. This constitutes an important tool for describing stacked structures.

The quasi-distance is the missing link between sets distance functions and a tool allowing the direct extraction of dimensional information on the homogeneous regions in greytone images. The efficiency of the distance function to generate segmentation markers is well known in the binary case [4]. Quasi-distance allows to extend this capability to greytone images. In fact, this operator performs many tasks at the same time: it is a filter which equalizes the homogeneous regions of the image; it quantifies the size of these homogeneous regions and finally, it enhances the most contrasted regions in the image, in a similar way a waterfalls algorithm acts. It is not so surprising that segmentations obtained with this operator are very close to those provided by the hierarchical segmentation by waterfalls. However, while the waterfalls algorithm proceeds by grouping regions, the use of quasi-distance leads directly to a similar result. One can say that, whereas the approach by waterfalls is a "bottom-up" approach, quasi-distance supplies at once a "top-down" hierarchical organization [2].

Finally, the extension of the notion of residue is relatively easy. Any transformation depending on a parameter can be used to generate residues as it has been shown for the hierarchical segmentation from piles. The residues can themselves be organized into a hierarchy, by selecting the residues of second, third order. This change of point of view brought with the residues is extremely fruitful.

8. References

[1] BEUCHER S., « Segmentation d'images et morphologie mathématique », Doctorate Thesis, Paris School of Mines, June 1990, 295 p.

[2] BEUCHER S., "Watershed, hierarchical segmentation and waterfall algorithm", Proc. Mathematical Morphology and its Applications to Image Processing, Fontainebleau, Sept. 1994, Jean Serra and Pierre Soille (Eds.), Kluwer Ac. Publ., Nld, 1994, pp. 69-76.

[3] BEUCHER S. & LANTUEJOUL C., "On the use of the geodesic metric in image analysis", Journal of Microscopy, Vol. 121, Part 1, January 1981, pp. 39-49.

[4] BEUCHER S. & LANTUEJOUL C., "Geodesic distance and image analysis", 5th

International Congress for Stereology, Salzburg, Austria, 3-8 Sept. 1979 - Mikroscopie, 37-1980, pp. 138-142.
[5] BEUCHER S. & VINCENT L., « Introduction aux outils morphologiques de segmentation », Journées ANRT déc. 1988, Traitement d'images en microscopie à balayage et en microanalyse par sonde électronique, ANRT, Paris 1990.
[6] BEUCHER S. & MEYER F., "Morphological segmentation", Journal of Visual Communication and Image Representation, n_ 1, Vol. 1, Oct. 1990.
[7] KRESCH R. & MALAH D., "Morphological reduction of skeleton redundancy", Signal Processing 38 (1994), September 1994, pp. 143-151.
[8] MARAGOS P. & SCHAFER R.W., "Morphological Skeleton Representation and Coding of Binary Images", IEEE Trans. ASSP, Vol. 34, No. 5, pp. 1228-1244, Oct 1986.
[9] MEYER F. & BEUCHER S., "The Morphological approach of segmentation: the watershed transformation", In Dougherty E. (Editor), Mathematical Morphology in Image Processing, Marcel Dekker, New York, 1992.
[10] SAPIRO G. & MALAH D., "Morphological image coding based on a geometric sampling theorem and a modified skeleton representation", Journal of Visual Communication and Image Representation, Vol.5, No. 1, March 1994, pp.29-40.